Problem 1: Show that for vectors \( v \in \mathbb{R}^n \) and \( \epsilon_{jk} \) the basis element of \( \mathfrak{so}(n) \) corresponding to an infinitesimal rotation in the \( j-k \) plane, one has
\[
e^{-\frac{1}{2} \gamma_j \gamma_k} \gamma(v) e^{\frac{1}{2} \gamma_j \gamma_k} = \gamma(e^{\theta \epsilon_{jk}} v)
\]
and
\[
[-\frac{1}{2} \gamma_j \gamma_k, \gamma(v)] = \gamma(\epsilon_{jk} v)
\]

Problem 2: Prove the following change of variables formula for the fermionic integral
\[
\int F(\xi) d\xi_1 d\xi_2 \cdots d\xi_n = \frac{1}{\det A} \int F(A\xi') d\xi_1' d\xi_2' \cdots d\xi_n'
\]
where \( \xi = A\xi' \), i.e.
\[
\xi_j = \sum_{k=1}^n A_{jk} \xi_k
\]
for any invertible matrix \( A \) with entries \( A_{jk} \).
For a skew-symmetric matrix \( A \), and \( n = 2d \) even, show that one can evaluate the fermionic version of the Gaussian integral as
\[
\int e^{\frac{1}{2} \sum_{j,k=1}^n A_{jk} \xi_j \xi_k} d\xi_1 d\xi_2 \cdots d\xi_n = Pf(A)
\]
where
\[
Pf(A) = \frac{1}{d! 2^{d}} \sum_{\sigma} (-1)^{|\sigma|} A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(n-1)\sigma(n)}
\]
Here the sum is over all permutations \( \sigma \) of the \( n \) indices. \( Pf(A) \) is called the Pfaffian of the matrix \( A \).

Problem 3: For the fermionic oscillator construction of the spinor representation in dimension \( n = 2d \), with number operator \( N_F = \sum_{j=1}^d a_{Fj}^\dagger a_{Fj} \), define
\[
\Gamma = e^{i\pi N_F}
\]
Show that
\[
\Gamma = \prod_{j=1}^d (1 - 2 a_{Fj}^\dagger a_{Fj})
\]
\[ \Gamma = c\gamma_1\gamma_2 \cdots \gamma_d \]
for some constant \( c \). Compute \( c \).

\[ \gamma_j \Gamma + \Gamma \gamma_j = 0 \]
for all \( j \).

\[ \Gamma^2 = 1 \]

\[ P_\pm = \frac{1}{2} (1 \pm \Gamma) \]
are projection operators onto subspaces \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) of \( \mathcal{H}_F \).

Show that \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) are each separately representations of \( \text{spin}(n) \) (i.e. the representation operators commute with \( P_\pm \)).

**Problem 4:** Use the fermionic analog of Bargmann-Fock to construct spinors in even dimensions as spaces of functions of anticommuting variables. Find the inner product on such spinors that is the analog of the one constructed using an integral in the bosonic case. Show that the operators \( a_{Fj} \) and \( a^*_{Fj} \) are adjoints with respect to this inner product.