Groups and Representations I: Problem Set 5

Due Monday, December 3

Problem 1: For the root system G_2 and the corresponding Lie algebra:

- Find the Cartan matrix.
- Explicitly construct the root system and show that it corresponds to the Dynkin diagram.
- Pick a consistent choice of positive roots, and display a basis of simple roots.
- For each simple root, construct the corresponding $\mathfrak{sl}(2, \mathbf{C})$ subalgebra and show how the full Lie algebra decomposes into irreducible representations of it.
- Find the fundamental weights and express them in terms of simple roots.
- Identify the lattice of integral weights, and the dominant Weyl chamber.
- Explicitly parametrize the set of finite dimensional irreducible representations.

Hint: You can consult chapter 22 of Fulton and Harris

Problem 2: Knapp, problems III.11-III-13.

Problem 3: Knapp, problem V.1

Problem 4: Knapp, problem V.11

Problem 5: Knapp, problem V.28

Problem 6: For the Lie algebra $L = \mathfrak{sl}(3, \mathbf{C})$, construct the Casimir operator and find its eigenvalues on each of the possible irreducible finite-dimensional representations of L.