Groups and Representations I: Problem Set 2

Due Monday, October 8

Problem 1: Prove that the set of right-invariant vector fields forms a Lie algebra under the Lie bracket operation, and show that it is isomorphic to T_eG. Define the diffeomorphism

$$\phi : g \in G \to \phi(g) = g^{-1} \in G$$

Show that if X is a left-invariant vector field, then $d\phi(X)$ is a right-invariant vector field, whose value at e is the same as that of $-X$. Show that

$$X \to d\phi(X)$$

gives an ismorphism of the Lie algebras of left and right invariant vector fields on G.

Problem 2: Classify the solvable Lie algebras of dimension 3 over $\mathbf{R}$, by doing problems 28-30 in Chapter I of Knapp.

Problem 3: Calculate the Killing bilinear from $B(\cdot,\cdot)$ for the cases of $L = \mathfrak{s}l(2,\mathbf{R})$ and $L = \mathfrak{s}l(2,\mathbf{C})$. What is the relation between these two Killing forms? Show that, in general, if $\mathfrak{g}$ is a complex Lie algebra, then the Killing form for $\mathfrak{g}^{\mathbf{R}}$ is given by twice the real part of the Killing form for $\mathfrak{g}$.

Problem 4: An automorphism of a Lie algebra L is an isomorphism of Lie algebras

$$a : L \to L$$

The set of such isomorphisms forms a group. Show that the Lie algebra of this group is the Lie algebra of derivations of L.

Problem 5: It was shown in class that if L is a Lie algebra of nilpotent endomorphisms of V, then there must be a non-zero $v \in V$ annihilated by all elements of L. Assuming this, complete the proof of Engel's theorem (Theorem 1.35 in Knapp) and prove that if each adX is nilpotent, then L is a nilpotent Lie algebra (corollary 1.38).

Give an example of a representation of a nilpotent Lie algebra L on a vector space V such that the elements of L cannot be represented by strictly upper triangular matrices.