Introduction and Motivation

Mathematics GR6434, Spring 2023

This semester we will be covering various topics in Lie groups, Lie algebras, and their representation theory, including

- The Heisenberg group and Stone-von Neumann theorem

  The Weil representation

  Relation to theta functions

  Analogy with Clifford algebra and the spinor representation

- Symplectic geometry and the orbit method

- Review of universal enveloping algebra, classification theory of complex Lie algebras

  Verma modules and highest-weight representations

  Harish-Chandra homomorphism

  Lie algebra cohomology and the Borel-Weil-Bott theorem (algebraic)

  Classification of finite-dimensional representations of semi-simple complex Lie algebras

- Geometric representation theory, starting with the Borel-Weil theorem

- Real semi-simple Lie groups and Lie algebras: classification

- $SL(2, \mathbf{R})$ and its representations, relation to modular forms

Some of this is conventional material for a course of this kind, but we'll emphasize a point of view sometimes called "the orbit philosophy". This has its roots in quantum mechanics and the idea of constructing representations by "geometric quantization". In addition, an attempt will be made to emphasize parts of the subject of relevance to number theory and the Langlands program.

1 Some background

The following notes give some background material that will be assumed or reviewed quickly:

- Background on Lie groups and Lie algebras

- Background on representations

- The simplest examples of quantum mechanical systems with one degree of freedom

2 Motivation from quantum mechanics

The fundamental postulates of quantum mechanics are:

- The state $\psi(t)$ of a physical system at a given time t is given by a vector in a complex Hilbert space $\mathcal{H}$. Recall that a Hilbert space is a vector space with an inner product $\langle\cdot,\cdot\rangle$, complete with respect to the metric defined by the inner product. Hilbert spaces may be finite or infinite dimensional as complex vector spaces.

- Observable quantities correspond to self-adjoint operators A on $\mathcal{H}$. The observable quantity corresponding to A will take the value a for states ψ_a that are eigenvectors of A with eigenvalue a (i.e. $A\psi_a = a\psi_a$).

- The most important observable is the energy and its corresponding self-adjoint operator is the Hamiltonian H. The time evolution of a system is given by the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = H\psi(t)$$

  where $\hbar$ is a constant (Planck's constant divided by 2π). We can choose our energy and/or time units so that $\hbar = 1$ and will generally do so.

To connect this formalism with experimentally observed numbers, one further postulates that measurements will satisfy

- The observed values of the observable corresponding to the operator A will be the eigenvalues of A.

- *Born Rule*: for a, ψ_a an eigenvalue and eigenfunction of A, in a state ψ one will observe the value a with probability $|\langle\psi,\psi_a\rangle|^2$ (here ψ, ψ_a are normalized to have norm 1).

The solution to the Schrödinger equation is formally given by

$$\Psi(t) = U(t)\psi(0), \quad U(t) = e^{-iHt}$$

Self-adjointness of H implies that $U(t)$ is unitary, so $\langle\psi,\psi\rangle$ is time-independent (and can thus be normalized to 1 for all times). For this formal solution to make sense we just need to be able to make sense of the exponential and show that it has the expected property

$$U(t_1)U(t_2) = U(t_1 + t_2)$$

For a bounded Hamiltonian operator H this is easy, you can just use the power series definition of the exponential. For unbounded operators some more serious analysis is required.

In the above situation the physical system is invariant under translations of the time variable (the Hamiltonian is time-independent), and the group $\mathbf{R}$ of time translations is represented on $\mathcal{H}$. The Hamiltonian operator is said to be a "generator" of the $\mathbf{R}$ symmetry:

$$t \to e^{-iHt}$$

is a continuous unitary representation of the additive group $\mathbf{R}$ on the Hilbert space $\mathcal{H}$. Stone's theorem (1930) tells us that there is a one-to-one correspondence between such representations and self-adjoint operators. This is actually a classification theorem for representations of $\mathbf{R}$.

Besides time translation symmetry, physical systems generally have other symmetries, by which one usually means group transformations which commute with the dynamics (action of time translations). Some examples include

- Translations in spatial directions, $G = (\mathbf{R}^3, +)$
- Rotations in space, $G = SO(3),\ SU(2)$.
- Lorentz transformations in special relativity, $G = SO(3,1),\ SL(2,\mathbf{C})$.
- Phase transformations of the wave-function, $G = U(1)$.
- U(N) transformations amongst N different kinds of particles. ("colors" or "flavors").
- S_n permutation transformations amongst n identical particles.

The Hilbert space of a quantum mechanical system will carry a unitary representation of any such symmetry groups of the physical system. Thus quantum mechanics produces interesting representations of all these groups and a sizable part of understanding a quantum mechanical system comes down to understanding the irreducible representations of these symmetry groups and how the state space (in particluar the energy eigenspaces) decomposes into irreducibles.

The relationship between quantum mechanics and representation is much deeper than just the use of symmetry groups that commute with the dynamics. We'll begin by discussing the Heisenberg group, for which the entire state space will be an irreducible representation. This representation can be thought of as a "quantization" of the classical phase space. We'll discuss in detail the "orbit philosophy" which posits that for any Lie group G, the G-orbits in $\mathfrak{g}^*$ can be thought of as generalized classical phase spaces, with "quantization" producing irreducible unitary representations of G.

3 Motivation from number theory

Much of number theory has to do with the Galois groups of number fields, in particular with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, the Galois group of the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. $\mathbf{Q}$ is an example of a global field, and it is useful to consider the corresponding local fields (the p-adic numbers $\mathbf{Q}_p$ for each prime p, $\mathbf{R}$ for the "infinite prime"), and the Galois groups $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ and $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \mathbf{Z}_2$.

These groups can be understood in terms of their representations, known as Galois representations. These are not Lie groups and the methods we'll develop in the course don't directly say anything about them. The Langlands program however posits a correspondence between Galois representations and Lie group representations. Very crudely, in the local field case, the Galois representations provide "Langlands parameters" which parametrize the irreducible Lie group (p-adic groups for finite p) representations. At the infinite prime, this local Langlands correspondence gives a parametrization of the infinite-dimensional irreducible representations of a real Lie group. If the instructor finally understands this well-enough, he'll try and at least give the statment of the real local Langlands correspondence later in the course.

A theme of the course will be that of finding the irreducible representations as subspaces of functions on a group. In the Langlands philosophy, representations of the Galois group of the global field $\mathbf{Q}$ are related to "automorphic representations", which come from the decomposition into irreducibles of functions on a product of the real Lie group G and all the corresponding p-adic groups for each p. One aspect of this decomposition is given by lookng at the decomposition into irreducibles of functions on a Lie group modulo a discrete group. For the case of $G = SL(2, \mathbf{R})$ one gets modular forms this way, and we will be explaining the relation between these and the representation theory of $SL(2, \mathbf{R})$ (and its double cover) which we will be discussing in detail.