Lie Groups and Representations, Spring 2016
Problem Set 7

Due Monday, March 21

Problem 1: For the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3)$, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ will be given by a polynomial algebra in two generators, one of which is given by the Casimir operator. Can you identify the second generator and show that it is invariant under the ρ-shifted action of the Weyl group?
Find the infinitesimal character of an arbitrary finite dimensional irreducible $\mathfrak{sl}(3)$ representation (i.e. how do the two generators of $Z(\mathfrak{g})$ act on the representation with highest weight $k_1\omega_1 + k_2\omega_2$ for ω_j the fundamental weights?

Problem 2: Consider the Koszul complex

$$0 \to U(\mathfrak{g}) \otimes \Lambda^n(\mathfrak{g}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} U(\mathfrak{g}) \otimes \Lambda^0(\mathfrak{g}) \xrightarrow{\epsilon} \mathbf{C} \to 0$$

where dim $\mathfrak{g} = n$ and

$$\begin{aligned}\partial(u \otimes X_1 \wedge \cdots X_k) = &\sum_{i=1}^{k} uX_i \otimes X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k \\ &+ \sum_{i<j} u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_k\end{aligned}$$

and ϵ projects to the constant term in $U(\mathfrak{g})$.
Show that the maps ∂ are $U(\mathfrak{g})$ homomorphisms, and satisfy $\partial \circ \partial = 0$.

Problem 3: Show that $H^1(\mathfrak{g}, \mathbf{C})$ vanishes for a semisimple Lie algebra.

Problem 4: Show that any non-zero element $[\omega] \in H^2(\mathfrak{g}, \mathbf{C})$ defines a new Lie algebra $\widetilde{\mathfrak{g}}$, given by

$$0 \to \mathbf{C} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where the Lie bracket on $\widetilde{\mathfrak{g}}$ is given by

$$[(X, c), (Y, d)] = ([X, Y], \omega(X, Y))$$