Lie Groups and Representations, Spring 2016
Problem Set 6

Due Monday, March 7

Problem 1: Show that the Casimir operator

$$C = \sum_i X_i X_i^*$$

is in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Here X_i is a basis, and X_i^* is the dual basis, with respect to the Killing form $B(\cdot, \cdot)$.

Problem 2: Show that C acts on an irreducible representation of $\mathfrak{g}$ with highest weight λ by multiplication by the scalar

$$||\lambda + \rho||^2 - ||\rho||^2$$

where the inner product is the one induced from restriction of the Killing form.

Problem 3: Use the Weyl character formula to prove the Kostant multiplicity formula, which says that

$$\mathrm{dim} V_\lambda(\mu) = \sum_{w \in W} P(w(\lambda + \rho) - (\mu + \rho))$$

where $P(\mu)$ is the number of way to write μ as a sum of positive roots.

Problem 4: Use the Weyl character formula to prove the Weyl dimension formula

$$\mathrm{dim} V_\lambda = \frac{\prod_{\alpha \in R^+} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \rho \rangle}$$

Use this to find the dimension of the representation of $\mathfrak{sl}(3, \mathbf{C})$ with highest weight $k_1 \omega_1 + k_2 \omega_2$, where ω_i are the fundamental weights.