Lie Groups and Representations, Spring 2016
Problem Set 4

Due Monday, February 22

Problem 1: For a simple complex Lie algebra $\mathfrak{g}$, the negative Weyl chamber C_- is just the $-C_+$, the negative of the dominant Weyl chamber. There will be a Weyl group element w_0 such that

$$w_0(C_+) = C_-$$

What is w_0 for $\mathfrak{sl}(n, \mathbf{C})$?
Show that if V is a finite-dimensional irreducible representation of $\mathfrak{g}$ with weights μ_j and highest weight λ, then the dual or contragredient representation of $\mathfrak{g}$ will have weights $-\mu_i$, and will be an irreducible representation with highest weight $-w_0(\lambda)$.

Problem 2: Find the fundamental weights ω_i for $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$. Show that the tensor product representations on the anti-symmetric tensor products $\Lambda^k(\mathbf{C}^n)$ for $k = 1, \ldots, n-1$ are irreducible representations with these highest weights.

Problem 3: Consider the fundamental weights corresponding a choice of simple roots of the Lie algebra $\mathfrak{g} = \mathfrak{so}(2n, \mathbf{C})$. Show that all but two of these are the highest weights of irreducible representations of $\mathfrak{so}(2n, \mathbf{C})$ on anti-symmetric tensor products $\Lambda^k(\mathbf{C})^{2n}$. For the other two fundamental weights, determine the dimensions of the representations with these highest weights and compute their characters.