Lie Groups and Representations, Spring 2016
Problem Set 10

Due Monday, April 18

Problem 1: Using the representation of n annihilation and creation operators $a_j, a_j^\dagger$ as differentiation and multiplication operators $\frac{\partial}{\partial z_j}, z_j$ on polynomials $\mathbf{C}[z_1, \cdots, z_n]$, find quadratic combinations of these operators that provide a representation of $\mathfrak{u}(n)$ on these polynomials. How does this representation decompose into irreducible sub-representations?

Problem 2: Recalling the realization of the symplectic Lie algebra $\mathbf{sp}(2n, \mathbf{R})$ in terms of quadratic polynomials in q_j, p_j from the last problem set, write the Weil representation in terms of operators Q_j, P_j, defined in terms of the $a_j, a_j^\dagger$. Show that the Weil representation restricts to the representation of Problem 1 on the sub Lie algebra $\mathfrak{u}(n) \subset \mathbf{sp}(2n, \mathbf{R})$. Finally show that this representation differs by a projective factor from the standard representation of $U(n)$ on homogeneous polynomials.

Problem 3: For the real Lie algebras $\mathfrak{h}_3$ and $\mathfrak{so}(3)$, explicitly find the co-adjoint orbits, together with the symplectic form on their tangent spaces.

Problem 4: Show that for a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on a complex vector space V, the Clifford algebra $\text{Cliff}(V.\langle\cdot,\cdot\rangle)$ is a filtered algebra, with associated graded algebra the exterior algebra $\Lambda^*(V)$.