Highest-weight Theory: Verma Modules

Math G4344, Spring 2012

We will now turn to the problem of classifying and constructing all finite-dimensional representations of a complex semi-simple Lie algebra (or, equivalently, of a compact Lie group). It turns out that such representations can be characterized by their "highest-weight". The first method we'll consider is purely Lie-algebraic, it begins by constructing a universal representation with a given highest-weight, then the finite-dimensional representation we want is found as a quotient of this. These universal representations are known as "Verma modules". They are infinite dimensional, not completely reducible, non-unitary and not representations of the corresponding group, but have quotients that do have the properties we want.

1 The $\mathfrak{sl}(2, \mathbf{C})$ case

This is a short review of the $\mathfrak{sl}(2, \mathbf{C})$ case, which you should have seen last semester.

One can take as a basis of $\mathfrak{sl}(2, \mathbf{C})$ elements e, f, h satisfying

$$[e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f$$

Representations decompose according to eigenvalues of h, with $V_\lambda \subset V$ the eigenspace of weight λ. e raises the weight

$$e : V_\lambda \to V_{\lambda+2}$$

and f lowers it

$$f : V_\lambda \to V_{\lambda-2}$$

By induction and use of the commutation relations one finds that

$$ef^{n+1} = f^{n+1}e + (n + 1)f^n(h - n)$$

In non-relativistic quantum physics, the theory of angular momentum operators is based on representation theory of the compact real form $\mathfrak{su}(2) = \mathfrak{spin}(3)$. Physicists choose rotations about the z-axis as their maximal torus. They name their operators giving a Lie algebra representation

$$\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$$

and choose these to be self-adjoint (so differing by i from mathematician's convention). J_3 is (up to a factor of 2) the operator representing our h. For computations they often complexify and work with our e, f as $J_1 \pm J_2$.

Representations V of $\mathfrak{sl}(2, \mathbf{C})$ are characterized by their highest-weight subspace, the $v \in V$ such that $ev = 0$. Irreducible repesentations have a one-dimensional highest weight subspace, and are characterized by the "highest weight", the h-eigenvalue of this subspace. An irreducible representation V^λ of highest weight λ can be constructed if you start with a $v \in V_\lambda$ and generate vectors

$$v, fv, f^2v, \cdots$$

that span V, giving a basis of one-dimensional subspaces of weights

$$\lambda, \lambda-2, \lambda-4. \cdots$$

If λ is negative or positive non-integral, this sequence never terminates and one gets an infinite dimensional representation. If $\lambda = n$ for some non-negative integer n, then the sequence will terminate, with f annihilating

$$f^n v$$

. This will give a finite-dimensional representation V^n of dimension $n+1$ with weights

$$n, n-2, \cdots, -n+2, -n$$

A physicist would describe V^n as the "spin $j = n/2$" representation.

The Weyl group $W = \mathbf{Z}_2$ just acts by changing the sign of a weight, leaving the set of weights of an irreducible invariant as it should.

One can reformulate the above construction of a representation as the sort of induced representation discussed earlier in this class. Consider the subalgebra $\mathfrak{b} \subset \mathfrak{sl}(2,\mathbf{C})$ generated by h and e. Start with a one-dimensional representation $\mathbf{C}_\lambda$ of $\mathfrak{b}$, with e acting trivially, h by λ and construct the induced representation

$$V(\lambda) = Ind_{\mathfrak{b}}^{\mathfrak{sl}(2,\mathbf{C})}\mathbf{C}_\lambda = U(\mathfrak{sl}(2,\mathbf{C})) \otimes_{\mathfrak{b}} C_\lambda$$

This is a Verma module with highest weight λ.

Recall that by the Poincaré-Birkhoff-Witt theorem as a vector space $\mathfrak{sl}(2,\mathbf{C})$ has a basis

$$f^i h^j e^k \quad i, j, k = 0, 1, 2, 3, \cdots$$

and the tensor product over $U(\mathfrak{b})$ means we identify

$$ab \otimes v_\lambda = a \otimes bv_\lambda$$

for $a \in \mathfrak{sl}(2,\mathbf{C})$, $b \in \mathfrak{b}$ and $v_\lambda \in \mathbf{C}_\lambda$, so

$$ah \otimes v_\lambda = a \otimes hv_\lambda = \lambda a \otimes v_\lambda$$

and

$$ae \otimes v_\lambda = a \otimes ev_\lambda = 0$$

As a vector space, $V(\lambda)$ is an infinite dimensional module with a basis

$$v_\lambda, fv_\lambda, f^2v_\lambda \cdots$$

The Casimir operator for $\mathfrak{sl}(2,\mathbf{C})$ is

$$C = \frac{1}{2}h^2 + ef + fe$$

Note that this is the normalization one gets using the inner product $(X, Y) = tr(XY)$ with the trace in the fundamental representation (2 by 2 complex matrices), a factor of 4 different than using the Killing form defined using the adjoint representation.

Since C is in the center of $Z(\mathfrak{sl}(2, \mathbf{C}))$ it will act by a scalar on irreducible modules, a scalar that one can compute by its action on a highest weight vector:

$$Cv_\lambda = (\frac{1}{2}h^2 + ef)v_\lambda = (\frac{1}{2}h^2 + (fe + h))v_\lambda = (\frac{1}{2}\lambda^2 + \lambda)v_\lambda$$

So C acts by $\frac{1}{2}\lambda^2 + \lambda$ on $V(\lambda)$. Note that this is invariant under the operation

$$\lambda \leftrightarrow -\lambda - 2$$

which is reflection about $\lambda = -1$ The center of $U(\mathfrak{sl}(2, \mathbf{C})))$ is the polynomial algebra $\mathbf{C}[C]$. Acting on Verma modules $V(\lambda)$ one gets polynomials in λ invariant under the above operation of shift by 1 and reflection. We'll encounter this peculiar sort of shift repeatedly in this subject. This identification of the center of the enveloping algebra with an algebra of polynomials invariant under a shift and Weyl reflection is a simple example of the Harish-Chandra isomorphism we will study in general later.

Hendrik Casimir, after whom the Casimir operator is named, was a physicist, whose study of the operator was motivated by its occurrence in this context in physics. There the operator

$$\mathbf{J}_1^2 + \mathbf{J}_2^2 + \mathbf{J}_3^2$$

is in the center of $U(\mathfrak{su}(2))$ and takes values $j(j+1)$ in the spin j (our $n/2$) representation.

For $\lambda < 0$ or positive non-integral, $V(\lambda)$ is irreducible. For λ non-negative integral, it is indecomposable, but has a sub-module $V(-\lambda - 2)$, with the same infinitesimal character. The finite dimensional irreducible module with highest weight λ appears here as the quotient

$$V^\lambda = V(\lambda)/V(-\lambda - 2)$$

2 The General Case

For a general complex simple Lie algebra $\mathfrak{g}$, with a choice of Cartan subalgebra $\mathfrak{h}$ positive roots, one has

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where

$$\mathfrak{n}^+ = \bigoplus_{+\ roots\ \alpha} \mathfrak{g}_\alpha$$

and

$$\mathfrak{n}^- = \bigoplus_{+\ roots\ \alpha} \mathfrak{g}_{-\alpha}$$

are nilpotent subalgebras. The Cartan subalgebra has dimension l, the rank of the Lie algebra, and all roots are integral combinations of l simple positive roots

$$\alpha_i \in \mathfrak{h}^*, \ i = 1, 2, \cdots, l$$

For each such α_i we get an $\mathfrak{sl}(2, \mathbf{C})$ subalgebra $\mathfrak{s}_{\alpha_i}$ of $\mathfrak{g}$. General representations of $\mathfrak{g}$ will be representations of each of these $\mathfrak{s}_{\alpha_i}$, and we can apply the results of the previous section.

For each i we can choose

$$e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}, h_i \in \mathfrak{h}$$

satisfying the relations

$$[e_i, f_j] = \delta_{ij} h_i$$
$$[h_i, e_j] = a_{ij} e_j$$
$$[h_i, f_j] == a_{ij} f_j$$

where

$$a_{ij} = \alpha_j(h_i)$$

are the entries of the so-called Cartan matrix and satisfy $a_{ii} = 2$. $h_i \in \mathfrak{h}$ is called the "co-root" associated to the root α_i.

For each i we have reflection maps

$$w_{\alpha_i}$$

that act by orthogonal transformations on $\mathfrak{h}^*$ (with respect to the invariant inner product), taking α_i to $-\alpha_i$. These generate the Weyl group W.

The co-roots h_i span an integral lattice in $\mathfrak{h}$, with the dual lattice spanned by the fundamental weights $\omega_i \in \mathfrak{h}^*$, which satisfy

$$\omega_i(h_j) = \delta_{ij}$$

For the $\mathfrak{sl}(2, \mathbf{C})$ case $\mathfrak{h}$ is one-dimensional, there is one positive root $\alpha \in \mathfrak{h}^*$ and one co-root $h \in \mathfrak{h}$, satisfying $\alpha(h) = 2$. The fundamental weight $\omega \in \mathfrak{h}^*$ satisfies $\omega(h) = 1$, so $\alpha = 2\omega$.

(should be a picture here)

We would like to classify all irreducible finite-dimensional representations V of a complex simple Lie algebra $\mathfrak{g}$, and will do this by finding its weights ω. These are the elements of $\mathfrak{h}^*$ we get from diagonalizing the $\mathfrak{h}$ action:

$$hv = \omega(h)v, v \in V_\omega \subset V, h \in \mathfrak{h}$$

We know the following about the weights of V:

- V will have a highest weight λ and highest weight-space V_λ of vectors $v \in V$ characterized by

  $$\mathfrak{n}^+ v = 0, hv = \lambda(h)v$$

  (otherwise repeated application of the e_i would give an infinite-dimensional module).

- The weights ω of V lie in the weight lattice of integral combinations of the fundamental weights ω_i (otherwise one would not get finite dimensionality under the action of the sub-algebra $\mathfrak{s}_{\alpha_i}$)

- The highest weight λ is "dominant", i.e. it satisfies

$$\lambda(h_i) \geq 0$$

  for each i (otherwise repeated application of e_i would give an infinite-dimensional module). The subspace of $\mathfrak{h}^*$ of elements non-negative on each of the h_i is called the "dominant Weyl chamber. Elements of the Weyl group permute choices of which roots are positive, and thus change the dominant Weyl chamber.

- The pattern of weights is invariant under the action so the Weyl group.

The classification theorem is

Theorem 1 (Highest-weight Theorem)**.** *Finite dimensional irreducible representations of a complex simple Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with integral dominant weights λ, which give the highest weight of the representation.*

We have seen that irreducible representations will have a highest weight, which must be an integral, dominant weight, but we have not shown that a finite dimensional representation with a given highest weight exists and is unique. One way to approach this is to generalize what we did for $\mathfrak{sl}(2, \mathbf{C})$ and to construct representations with a given highest weight by induction in the Lie algebra, these are the Verma modules:

Definition 1 (Verma Module)**.** *For a complex simple Lie algebra $\mathfrak{g}$ with Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, the Verma module with highest weight $\lambda \in \mathfrak{h}^*$ is*

$$V(\lambda) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbf{C}_\lambda$$

Here $\mathbf{C}_\lambda$ is the one-dimensional $\mathfrak{b}$ module where $\mathfrak{n}^+$ acts trivially, $\mathfrak{h}$ by weight λ.

As in the $\mathfrak{sl}(2, \mathbf{C})$ case these modules will be infinite-dimensional and indecomposable. When λ is dominant integral, they will have maximal proper sub-modules M such that $V(\lambda)/M$ is finite-dimensional. The proof of this is somewhat involved, see for example [1],[2] or [3]. We will try and partially make up for the lack of a proof of finite dimensionality when we turn to another way of constructing these representations, using induction in the group, where finite-dimensionality follows from general theorems in either analysis or algebraic geometry.

The full story of how Verma modules fit together as sub-modules of each other, providing finite-dimensional modules as quotients is rather complicated and not so easily proved. It is

Theorem 2 (Bernstein-Gelfand-Gelfand Resolution)**.** *There is an exact sequence of Verma modules*

$$0 \to V(w_0 \cdot \lambda) \to \cdots \bigoplus_{w \in W, l(w)=k} V(w \cdot \lambda) \to \cdots \to V(\lambda) \to V^\lambda \to 0$$

where

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

$l(w)$ is the length of the Weyl group element w, w_0 is the Weyl group element of maximal length. Here ρ is half the sum of the positive roots.

The subject of "geometric representation theory" relates Verma modules to the geometry of the flag variety G/B. B acts on the left, with a finite number $|W|$ of B-orbits (this is the "Bruhat decomposition"). For $G = SL(2, \mathbf{C})$, $G/B = \mathbf{C}P^1$ and there are two B-orbits: the Riemann sphere minus the South pole, and the South pole, of dimension 2 and 0 respectively. Verma modules correspond to delta-function distributions on the B-orbits, dual Verma modules to holomorphic functions on the orbits, singular at the boundary. For more details, one source is [4]. For a general introduction to geometric representation theory techniques, see [5].

Next time we'll discuss Borel-Weil theory, which provides another sort of construction of these finite-dimensional representations by induction, using "holomorphic induction" in the group. There is a third possible way to construct all irreducibles, which requires a separate analysis though for each class of Lie algebras. Here one begins by somehow constructing the fundamental representations then taking tensor powers. The highest weight of the tensor product of two representations is the sum of the highest weights of the two representations, so we can get all dominant integral highest weights by tensoring fundamental representations. These tensor products will not themselves be irreducibles, but will contain the irreducibles we want.

3 The Casimir

Recall that for a complex semi-simple Lie algebra $\mathfrak{g}$ we have a non-degenerate inner product $B(\cdot, \cdot)$, which can be taken to be the Killing form

$$B(X, Y) = tr(ad(X)ad(Y))$$

(or we could use the trace in another representation). If X_i are an orthonormal basis of $\mathfrak{g}$ with respect to B, the Casimir element is a quadratic element in $U(\mathfrak{g})$ given by

$$C = \sum_i X_i^2$$

This is in the center $Z(\mathfrak{g})$ so will act as a scalar on an irreducible representation. One of your assignments in the next problem set will be to compute this scalar in terms of the highest weight of the representation. The answer is:

Claim: C acts on the irreducible highest weight representation V^λ by

$$Cv = (<\lambda, \lambda> + 2 <\lambda, \rho>)v = (<\lambda + \rho, \lambda + \rho> - <\rho, \rho>)v$$

In the exercises you will also show that this scalar is invariant under the transformation

$$\lambda \to w(\lambda + \rho) - \rho$$

for any element w in the Weyl group. This generalizes the invariance under

$$\lambda \leftrightarrow -\lambda - 2$$

that we saw in the $\mathfrak{sl}(2, \mathbf{C})$ case. We will see later in our discussion of the Harish-Chandra isomorphism that this generalizes to other elements in the center, all of which will act by scalars invariant under this shifted-Weyl action.

Note that, starting a Verma module $V(\lambda)$ for a dominant integral weight λ, this gives us $|W|$ Verma modules with the same infinitesimal character, exactly the ones that occur in the BGG resolution.

4 Examples

The simplest class of examples to work out is that of the Lie algebras $\mathfrak{sl}(n, \mathbf{C})$. Here one can work with n by n complex, trace-free matrices M, and make the following choices:

- The Cartan subalgebra is $\mathfrak{h}$= diagonal matrices H with trace 0. Let $e_i \in \mathfrak{h}^*$ be the element that takes a matrix to its i'th diagonal entry:

  $$e_i(H) = H_{ii}$$

  The trace-free condition implies that $\sum_{i=1}^n e_i = 0$

- Let E_{ij} be the matrix M with $M_{ij} = 1$, all other elements 0. These are eigenvectors for the adjoint action of $h =\in \mathfrak{h}$:

  $$ad(h)E_{ij} = [H, E_{ij}] = (H_{ii} - H_{jj})E_{ij} = (e_i - e_j)(H)E_{ij}$$

- The roots are thus given by

  $$\alpha_{i,j} = e_i - e_j, \; i \neq j$$

  and the root spaces are

  $$\mathfrak{g}_{\alpha_{i,j}} = \mathbf{C}E_{ij}$$

- A standard choice of positive roots is to choose the α_{ij} with $i < j$. Then $\mathfrak{n}^+$ is the Lie algebra of strictly upper triangular n by n matrices and $\mathfrak{n}^-$ is the Lie algebra of strictly lower triangular n by n matrices.

- With this choice of positive roots, a choice of the $n-1$ simple roots is the $\alpha_{i,i+1}$, i.e.

$$e_1 - e_2, e_2 - e_3, \cdots e_{n-1} - e_n$$

- The fundamental weights then are:

$$\omega_1 = e_1, \omega_2 = e_1 + e_2, \cdots, \omega_{n-1} = e_1 + e_2 + \cdots + e_{n-1}$$

  The representation with highest weight e_1 is the defining representation on $\mathbf{C}^n$. The fundamental representation with highest weight ω_k is the wedge-product representation on $\Lambda^k(\mathbf{C}^n)$. The representation with highest weight ke_1 is the representation on the symmetric product $S^k(\mathbf{C}^n)$, and for $\mathfrak{sl}(2,\mathbf{C})$ these are all the irreducibles, on homogeneous k-th order polynomials . A general irreducible can thus be constructed out of anti-symmetric and symmetric products of $\mathbf{C}^n$. The subject of "Schur-Weyl" duality relates representations of the symmetric group and representations of $\mathfrak{sl}(n,\mathbf{C})$.

- The Weyl group is $W = S_n$, generated by $n-1$ transpositions s_i that take

$$e_i \leftrightarrow e_{i+1}, i = 1, 2, \cdots, n-1$$

- The dominant integral weights are the

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{n-1} e_{n-1}$$

  with

$$\lambda_i \in \mathbf{Z}^+, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0$$

- $$\rho = \frac{1}{2}\sum_{i<j} \alpha i, j = ne_1 + (n-1)e_2 + \cdots + e_{n-1}$$

For the case $n = 3$ one can draw the weight lattice by starting with three unit vectors e_i, with $\frac{2\pi}{3}$ between any two of them. Then, since $e_3 = -e_1 - e_2$, all weights can be written as integral linear combinations of the e_1 and e_2. In particular one has

- $\omega_1 = e_1$, this is the highest weight of the defining representation on $\mathbf{C}^3$. The other two weights in this representation are e_2 and e_3
- $\omega_2 = e_1 + e_2$, this is the highest weight of the representation $\Lambda^2\mathbf{C}^3$.
- Non-negative integral combinations of ω_1 and ω_2 give all the highest weights of irreducibles.
- $\alpha_{1,2} = e_1 - e_2$ is one of the simple roots, a weight that is not dominant.
- $\alpha_{2,3} = e_2 - e_3$ is the other simple root, also a non-dominant weight.

- The third positive root is

$$\alpha_{1,3} = e_1 - e_3 = \alpha_{1,2} + \alpha_{2,3} = \rho = \frac{1}{2}(\alpha_{1,2} + \alpha_{2,3} + \alpha_{1,3})$$

  It is half the sum of the positive roots, and the highest weight of the adjoint representation.

- The Weyl group is $W = S_3$, generated by two elements: $s_{\alpha_{1,2}}$ (reflection about line perpendicular to $\alpha_{1,2}$) and $s_{\alpha_{2,3}}$ (reflection about line perpendicular to $\alpha_{2,3}$). There are six elements of the Weyl group: 1, of length 0, $s_{\alpha_{1,2}}, s_{\alpha_{2,3}}$ of length 1, $s_{\alpha_{1,2}}s_{\alpha_{2,3}}, s_{\alpha_{2,3}}s_{\alpha_{1,2}}$ of length 2, and $s_{\alpha_{1,2}}s_{\alpha_{2,3}}s_{\alpha_{1,2}}$ of length 3.

- The BGG resolution of a highest weight representation in terms of Verma modules is

$$0 \to V(s_{\alpha_{1,2}}s_{\alpha_{2,3}}s_{\alpha_{1,2}}(\lambda+\rho)-\rho) \to V(s_{\alpha_{1,2}}s_{\alpha_{2,3}}(\lambda+\rho)-\rho) \oplus V(s_{\alpha_{2,3}}s_{\alpha_{1,2}}(\lambda+\rho)-\rho)$$
$$\to V(s_{\alpha_{1,2}}(\lambda+\rho)-\rho) \oplus V(s_{\alpha_{2,3}}(\lambda+\rho)-\rho) \to V(\lambda) \to V^\lambda \to 0$$

- The flag variety $SL(3,\mathbf{C})/B = SU(3)/(U(1)\times U(1))$ is a six real-dimensional manifold, also a complex 3-dimensional Kähler manifold. The B action from the left decomposes it into six B-orbits, which can be identified with a $\mathbf{C}^3$. two copies of $\mathbf{C}^2$ two copies of $\mathbf{C}$ and a point.

Should be two pictures here, one showing the weights described above, the other the six Weyl chambers.

In the discussion section, Alex Ellis will go over some of the details of what the weight diagrams for various representations look like in the $\mathfrak{sl}(3,\mathbf{C})$ case.

References

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[4] Segal, G., *Loop Groups*

[5] Oxford University Press, 1985. Chapter 14.5. Hotta, R., Takeuchi, K., and Tanisaki, T., *D-Modules, Perverse Sheaves, and Representation Theory* Birkhauser, 2008.