Groups and Representations, Spring 2012
Problem Set 4
Due Wednesday, April 18

Problem 1:

1. Show that the Clifford algebra is a filtered algebra, with associated graded algebra the exterior algebra.

2. Show that the quotient

$$U_c(\mathfrak{h}(n)) = U(\mathfrak{h}(n))/(C - c\mathbf{1})$$

is also a filtered algebra, with associated graded algebra a polynomial algebra.

3. Show that this filtration $F^i(U_c(\mathfrak{h}(n)))$ satisfies

$$[F^i, F^j] \subset F^{i+j-2}$$

that F^1 is isomorphic to the Heisenberg algebra, and F^2 is isomorphic to a semi-direct product of the Heisenberg algebra and another Lie algebra (what is the dimension of this Lie algebra?)

Problem 2: If e_i are a basis of the Clifford algebra $C(n)$, show that the $\frac{1}{2}e_ie_j$, for $i < j$, satisfy the same commutation relations as the standard basis $L_{ij} = E_{ij} - E_{ji}$ elements for the antisymmetric n by n matrices. This shows that $Lie(Spin(n))$ is isomorphic to $Lie(SO(n))$.

Problem 3: Using the Clifford algebra, derive formulae for

1. The action of the Lie algebra of $Spin(n)$ on vectors, i.e for infinitesimal rotations in the $i - j$ plane.

2. The action of the group $Spin(n)$ on vectors, i.e. the formula for a rotation by angle θ in the $i - j$ plane.

Problem 4: The $\frac{1}{2}e_{2j-1}e_{2j}$ provide a basis of a Cartan subalgebra in $Lie(Spin(2n))$. Choose a complex structure on $\mathbf{R}^{2n}$ that provides an identification with $\mathbf{C}^n$. Use this to relate the Clifford algebra and the CAR algebra corresponding to a basis of $\mathbf{C}^n$.
Use this CAR algebra to construct explicitly a Lie algebra representation of $Lie(Spin(2n))$ on $\Lambda^*(\mathbf{C}^n)$. Find the weights of this representation, show that it is the sum of two irreducible representations.

Problem 5: Using the definition of the Bargmann-Fock space given in class, show that the operators $a_i^\dagger$ and a_i corresponding to a basis of $\mathbf{C}^n$ are adjoint operators on this space.