Groups and Representations, Spring 2012
Problem Set 2
Due Monday, February 27

Problem 1: Show that for a simple Lie algebra $\mathfrak{g}$ the Casimir element $C \in U(\mathfrak{g})$ acts on an irreducible highest weight representation with highest weight λ as multiplication by the scalar

$$< \lambda + \rho, \lambda + \rho > - < \rho, \rho >$$

where ρ is half the sum of the positive roots.
Show that this scalar is invariant under the shifted Weyl group action

$$\lambda \to w \cdot \lambda = w(\lambda + \rho) - \rho$$

Problem 2: Show that the dual $(V^\lambda)^*$ of a finite dimensional highest weight representation of $\mathfrak{g}$ has highest weight $-w_0\lambda$, where w_0 is the Weyl group element of maximal length, which takes elements in the dominant Weyl chamber to the anti-dominant Weyl chamber (i.e. changes the sign).

Problem 3: For $\mathfrak{g} = \mathfrak{sl}(3)$ use the Weyl integral formula to find the dimensions of the representations with highest weight

$$n_1\omega_1 + n_2\omega_2$$

where ω_1, ω_2 are the fundamental weights.
Pick one of these representations with $n_1 > 1$ and $n_2 > 1$ and draw its weight diagram, showing also the fundamental weights and the roots.

Problem 4: Show that the Verma module $V(\lambda)$ has character (this is a formal series, since this is an infinite dimensional representation)

$$\chi_{V(\lambda)} = \frac{e^\lambda}{\prod_{\alpha \in R^+}(1 - e^{-\alpha})}$$

Assuming the BGG resolution of a finite-dimensional representation V^λ with highest-weight λ, prove the Weyl character formula using Verma modules.

Problem 5: In class we defined the Kostant partition function $P(\mu)$ as the number of ways one can write the integral weight μ as an integral combination of the positive roots. Use the Weyl character formula to prove the Kostant multiplicity formula, which says that a weight μ occurs in a the highest weight representation V^λ with multiplicity

$$\sum_{w \in W} (-1)^{l(w)} P(w(\lambda + \rho) - (\mu + \rho))$$

Problem 6: Prove Steinberg's formula: the multiplicity of V^λ in the decomposition of the tensor product $V^\mu \otimes V^\nu$ is

$$\sum_{w, w' \in W} (-1)^{l(w)l(w')} P(w(\lambda + \rho) + w'(\mu + \rho) - \nu - 2\rho)$$