Harmonic Analysis on Compact Lie Groups: the Peter-Weyl Theorem

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1 Peter-Weyl as a special case of induction

Last time we discussed induction for Lie group representations. For a pair of Lie groups $H \subset G$ and a representation W of H, the induced G representation is

$$Ind_H^G(W) = Map_H(G, W) = \{f : G \to W : f(gh) = (\pi(h^{-1})f(g)) \ \forall \, g \in G, h \in H\}$$

with G acting by

$$\pi(g)f(g_0) = f(g^{-1}g_0)$$

and it satisfies the Frobenius reciprocity relation

$$Hom_G(V, Ind_H^G(W)) = Hom_H(V, W)$$

Now, consider the special case of $H = \mathbf{1}$, the trivial subgroup, with $W = \mathbf{C}$, the trivial representation. Then

$$Ind_{\mathbf{1}}^G \mathbf{C} = Map(G, \mathbf{C})$$

and Frobenius reciprocity says

$$Hom_G(V, Map(G, \mathbf{C})) = Hom_{\mathbf{C}}(V, \mathbf{C}) = V^*$$

So, the space of functions on G decomposes into irreducibles V_i with multiplicity of V_i just $dim\ V_i^* = dim\ V_i$.

The space $Maps(G, \mathbf{C})$ actually carries two commuting actions of G. The one we have been considering so far is the left action

$$\pi_L(g)f(g_0) = f(g^{-1}g_0)$$

but there is also a right action

$$\pi_R(g)f(g_0) = f(g_0 g)$$

and the two actions commute. The way $Maps(G, \mathbf{C})$ decomposes into irreducibles under these two separate actions is an example of a general principle. If a representation (π_1, V) of G_1 decomposes into irreducibles V_i, and the same V carries a commuting representation (π_2, V) of a group G_2 (which may or may not be the same as the group G_1), then the spaces of intertwining operator $Hom_{G_1}(V_i, V)$ (also called "multiplicity spaces") provide representations of G_2. In this case the groups are identical and all you get by this construction is the dual representation.

The Peter-Weyl theorem will tell us that

$$Map(G, \mathbf{C}) = \bigoplus_{i \in Irreps(G)} V_i \otimes V_i^*$$

as a representation of $G \times G$, with the first G acting on $Map(G, \mathbf{C})$ by the left regular representation, the second by the right regular representation. The first G acts on the first factor in each term of the sum, the second G acts on the second factor. Note that the theorem does not tell us either what the set of irreducible representatives $Irrep(G)$ is, or how to characterize or construct the irreducibles V_i.

So far we have carefully avoided specifying what space of functions $Map(G, \mathbf{C})$ we are talking about. This subject is in some sense a generalization of Fourier analysis (the case $G = U(1)$), and so is sometimes called "non-abelian harmonic analysis". For compact Lie groups, the analytic issues are easily dealt with, but become much trickier in the non-compact case. The Peter-Weyl theorem says that representations of compact Lie groups behave very much like representations of finite groups, with the analytic issues similar to those that occur for Fourier series. So, we'll start by quickly reviewing these two subjects.

2 Fourier Series

For Fourier series, two important function spaces to consider are

- Trigonometric polynomials: sums of the form

  $$f(\theta) = \sum_{n=-\infty}^{n=+\infty} a_n e^{in\theta}$$

  with a finite number of non-zero terms.

- $L^2(S^1)$: Lebesque square-integrable functions on $U(1) = S^1$ (or, periodic functions on $\mathbf{R}$ with period 2π). This is an inner-product space with inner-product

  $$< f, g >= \frac{1}{2\pi} \int_0^{2\pi} \overline{f} g$$

The first space is dense in the second, and the $e^{in\theta}$ give an orthonormal basis. Note that the first space carries an action of the Lie algebra and is more tractable algebraically, the second is complete and more tractable analytically.

We can interpret this in terms of representation theory of $G = U(1)$. Group elements are given explicitly by $g = e^{i\theta}$ and group multiplication is just complex multiplication. Since the group is Abelian, all irreducible representations are one-dimensional. They are indexed by an integer n and given by

$$\pi_n(e^{i\theta}) = e^{in\theta} \in U(1) \subset GL(1, \mathbf{C})$$

$U(1)$ representations are completely reducible, since one can average over $U(1)$ any inner product on a representation to get an invariant one.

The characters of the irreducible representations are just

$$\chi_n(e^{i\theta}) = e^{in\theta}$$

The space of characters is the space of all functions on the group, it is Abelian so all functions are conjugation invariant. The characters are orthonormal with respect to the invariant integral on the group:

$$< \chi_n, \chi_m >= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{im\theta} = \begin{cases} 0 & if\ n \neq m \\ 1 & if\ n = m \end{cases}$$

The analog of the (left) regular representation here is the action on $L^2(S^1)$ given by translation:

$$\pi(e^{i\theta})f(\theta_0) = f(\theta_0 - \theta)$$

(the analog of the right regular representation is essentially the same, except shifting by a positive angle, so there's not much use in considering, $U(1) \times U(1)$, i.e. both the right and left actions in this case.) The special case of Peter-Weyl here says that

$$Maps(U(1), \mathbf{C}) = \bigoplus_{n \in \mathbf{Z}} V_n$$

where the irreducible representations are labeled by $n \in \mathbf{Z}$ and given explictly as the 1-dimensional spaces of functions

$$V_n = \{f(\theta_0) : f(\theta_0 - \theta) = e^{in\theta}f(\theta_0)\} = \mathbf{C}e^{-in\theta_0}$$

Here one can interpret $Maps(U(1), \mathbf{C})$ as finite trigonmetric polynomials with the standard direct sum, or as $L^2(S^1)$ in which case the direct sum has to be interpreted as a "completed" direct sum, allowing infinite sequences, as long as the L^2 norm is finite.

3 Convolution

For the case $G = U(1)$ we know the irreducible representations and their characters. Given an arbitrary $U(1)$ representation, we would like to be able to decompose it into these irreducibles

There is another interesting product that one can define on $L^2(S^1)$, besides the usual point-wise multiplication:

Definition 1 (Convolution)**.** *The convolution of two functions f_1 and f_2 in $L^2(S^1)$ is the function*

$$f_1 * f_2 = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta - \theta')f_2(\theta')d\theta'$$

This product is commutative and associative (the generalization we will see later will be commutative only for commutative groups).

One use of the convolution product is to construct an orthogonal projection

$$L^2(S^1) \to V_n$$

using convolution with an irreducible character. It is easy to show that if

$$f = \sum_n a_n e^{in\theta}$$

then

$$f * \chi_n = a_n e^{in\theta}$$

and characters provide idempotents in the algebra $(L^2(S^1), *)$, satisfying

$$\chi_n * \chi_m = \begin{cases} 0 & if\ n \neq m \\ \chi_n & if\ n = m \end{cases}$$

We have

$$f = \sum_n f * \chi_n$$

The construction of the convolution, with the same properties as above, generalizes to the case of non-abelian groups using the invariant Haar integral $\int_G$:

Definition 2. *The convolution product on $L^2(G)$ is*

$$(f_1, f_2) \to f_1 * f_2 = \int_G f_1(gh^{-1}) f_2(h) dh$$

With this product, the functions on G become an algebra, called the group algebra, which is non-commutative when the group is non-commutative. Given a representation (π, V) of G, one can make V into a module over the group algebra, defining an algebra homomorphism

$$\tilde{\pi} : L^2(G) \to End(V)$$

by

$$\tilde{\pi} v = \int_G f(g) \pi(g) v dg$$

One can check that this satisfies

$$\tilde{\pi}(f_1 * f_2) = \tilde{\pi}(f_1) \tilde{\pi}(f_2)$$

An alternate approach to representation theory of groups is to think of it as the theory of these algebras and their modules.

4 Finite Groups

For finite group representations the group algebra is $\mathbf{C}G = Map(G \to \mathbf{C})$, with no question about what this means since it is a finite spaces of maps. The convolution is the same as in the Lie group case with

$$\int_G \to \frac{1}{|G|} \sum_{g \in G}$$

One has the following facts:

- The group algebra is a sum of matrix algebras (by Wedderburn's theorem)

  $$\mathbf{C}G = \oplus_i M(n_i, \mathbf{C}) = \oplus_i End(V_i) = \oplus_i (V_i^* \otimes V_i)$$

  The sum is over the finite number of irreducible representations of G, with n_i the dimension of the i'th irreducible.

- The characters χ_{V_i} of the irreducible representations provide an orthonormal basis of $\mathbf{C}G^G$, the conjugation invariant functions on G.

- The subspaces of $\mathbf{C}G$ corresponding to $End(V_i) = V_i^* \otimes V_i$ is the $(dim\ V_i)^2$ subspace of matrix elements of the representation (π_i, V_i). To $X \in End(V_i)$ one associates the function

  $$tr(\pi_i(g)X)$$

  or equivalently, to $l \in V_i^*, v \in V_i$ one associates

  $$l(\pi_i(g)v)$$

- The subspaces corresponding to irreducibles are orthogonal, with orthogonal projection from $\mathbf{C}G$ onto the $End(V_i)$ subspace given by

  $$f \to f * \chi_{V_i}$$

The proofs of these facts just use Schur's lemma, exploiting the fact that

$$Hom_G(V_i, V_j) = \emptyset, i \neq j$$

A good source for these proofs is Constantin Teleman's notes[4].

5 Compact Lie Groups and the Peter-Weyl Theorem

For compact Lie groups, one can proceed as for finite groups, just changing

$$\frac{1}{|G|} \sum_{g \in G} \to \int_G$$

and derive the Peter-Weyl theorem, which is essentially the statement that everything works in the compact Lie group case the same way as in the finite group case:

Theorem 1 (Peter-Weyl)**.** *The matrix elements of finite dimensional irreducible representations form a complete set of orthogonal vectors in $L^2(G)$.*

Equivalently, this theorem says that every $f \in L^2(G)$ can be written uniquely as a series

$$f = \sum_i f_i,\ \ f_i \in End(V_i)$$

which we can also write

$$L^2(G) = \hat{\oplus}_i End(V_i) = \hat{\oplus}_i (V_i^* \otimes V_i)$$

where $\hat{\oplus}$ is a completed direct sum.

There's also an easy corollary, which says that one can expand any conjugation invariant function in terms of characters of irreducible representations:

Corollary 1. *The characters of finite dimensional irreducible reps of G give an orthonormal basis of $L^2(G)^G$ (the conjugation invariant subspace of $L^2(G)$).*

The one tricky part of the Peter-Weyl theorem is that one has to show that finite-dimensional representations don't miss anything. One could in principle have an infinite dimensional irreducible representation of G whose matrix elements would be in $L^2(G)$, but orthogonal to all matrix elements of finite-dimensional representations. To show that this doesn't happen, one can proceed as follows:

Assume the existence of an $f \in L^2(G)$ that is orthogonal to all finite dimensional sub-representations in $L^2(G)$. Construct a sequence of functions K_n on G satisfying

$$\overline{K_n(g^{-1})} = K_n(g)$$

approaching the δ-function at the identity as $n \to \infty$. The convolution operator

$$T_{K_n} : f \to f * K_n$$

is a compact self-adjoint operator on $L^2(G)$ and by the spectral theorem $ker(T_{K_n})^\perp$ is a direct sum of finite-dimensional eigen-spaces, so orthogonal to f. So $f \in ker\ K_n$ for each n and in the limit $n \to \infty$ one sees that $f = 0$.

For a more detailed proof of Peter-Weyl, see Terry Tao's blog entry[1] on the subject. Chapter 9 of [2] contains an extensive discussion of the various equivalent forms of the theorem. Chapter 3 of [3] provides a much more detailed discussion of harmonic analysis on compact Lie groups, with a proof of Peter-Weyl in section 3.3. For a good discussion of the finite group case, see Constantin Teleman's notes[4].

References

[1] http://terrytao.wordpress.com/2011/01/23/the-peter-weyl-theorem-and-non-abelian-fourier-analysis-on-compact-groups

[2] Segal, G., Lie Groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.

[3] Sepanski, M., *Compact Lie Groups,* Springer-Verlag, 2006.

[4] *Notes for a course on representation theory, 2005.* http://math.berkeley.edu/~teleman/math/RepThry.pdf