Induced Representations and Frobenius Reciprocity

Math G4344, Spring 2012

1 Generalities about Induced Representations

For any group G and subgroup H, we get a restriction functor

$$Res_H^G : Rep(G) \to Rep(H)$$

that gives a representation of H from a representation of G, by simply restricting the group action to H. This is an exact functor. We would like to be able to go in the other direction, building up representations of G from representations of its subgroups. What we want is an induction functor

$$Ind_H^G : Rep(H) \to Rep(G)$$

that should be an adjoint to the restriction functor. This adjointness relation is called "Frobenius Reciprocity".

Unfortunately, there are two possible adjoints, Ind could be a left-adjoint

$$Hom_G(Ind_H^G(V), W) = Hom_H(V, Res_H^G(W))$$

or a right-adjoint. Let's call the right-adjoint "coInd"

$$Hom_G(V, coInd_H^G(W)) = Hom_H(Res_H^G(V), W)$$

Given an algebra A, with sub-algebra B, tensor product $\otimes$ and Hom of modules provide such functors. Since

$$Hom_A(A \otimes_B V, W) = Hom_B(V, Res_B^A(W))$$

we can get a left-adjoint version of induction using the tensor product and since

$$Hom_A(V, Hom_B(A, W)) = Hom_B(Res_B^A(V), W)$$

we can get a right-adjoint version of induction using Hom. So

Definition 1 (Induction for finite groups)**.** *For H a subgroup of G, both finite groups, and V a representation of H, one has representations of G defined by*

$$Ind_H^G(V) = \mathbf{C}G \otimes_{\mathbf{C}H} V$$

and

$$coInd_H^G(V) = Hom_{\mathbf{C}H}(\mathbf{C}G, V)$$

and

Theorem 1 (Frobenius reciprocity for finite groups)**.** *Ind_H^G is a left-adjoint functor to the restriction functor, $coInd_H^G$ is a right-adjoint.*

It turns out that for finite groups, the Ind and $coInd$ functors are isomorphic, so you can use either one to taste. An important special case to consider is induction from the trivial representation, where Frobenius reciprocity says

$$Hom_G(CG \otimes_{\mathbf{C}H} \mathbf{C}, V) = Hom_H(\mathbf{C}, V) = (Hom_{\mathbf{C}}(\mathbf{C}, V))^H = V^H$$

which implies that the multiplicity of V in $Ind_H^G\mathbf{C}$ is given by the dimension of the H-invariant subspace of V. So, in general induction from the trivial representation will give a reducible representation with an interesting endomorphism algebra, a Hecke algebra:

Definition 2 (Hecke algebra for pair of finite groups)**.** *Given a pair of finite groups $H \subset G$, one can define the Hecke algebra*

$$Hecke(G, H) = Hom_G(Ind_H^G\mathbf{C}, Ind_H^G\mathbf{C}) = (Ind_H^G\mathbf{C})^H$$

Note that for any G representation V this algebra acts on

$$V^H = Hom_G(Ind_H^G\mathbf{C}, V)$$

For some choices of H and G, for example $G = GL(n, \mathbf{F}_q)$, H the Borel subgroup of upper triangular matrices, this allows one to study representation of the algebra $\mathbf{C}G$ in terms of a much simpler algebra $Hecke(G, H)$ (which is commutative in this case), at least for representations such that $V^H \neq 0$.

Co-induction from the trivial representation gives

$$coInd_H^G\mathbf{C} = Hom_{\mathbf{C}H}(\mathbf{C}G, \mathbf{C})$$

which is the space of functions on G, right-invariant under H (G acts on the left). Here the Hecke algebra is

$$(coInd_H^G\mathbf{C})^H$$

which is the convolution algebra of H-biinvariant functions on G. It acts on

$$Hom_H(V, \mathbf{C}) = (V^*)^H$$

2 Induction for Lie algebras

To make life confusing, it turns out that the first of these is what is most useful for Lie algebras, whereas it's the second that is most useful for Lie groups. So, for us, induction for Lie algebra representations will be the first version, for Lie groups something like the second. You just need to be careful to use the proper version of Frobenius reciprocity (left-adjoint to restriction for Lie algebras, right-adjoint to restriction for Lie groups).

So, for Lie algebras

Definition 3. *Given a Lie algebra* $\mathfrak{g}$ *with subalgebra* $\mathfrak{h}$*, if* V *is a representation of* $\mathfrak{h}$*, the induced representation of* $\mathfrak{g}$ *is*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$$

This satisfies the Frobenius reciprocity relation

$$Hom_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, W) = Hom_{\mathfrak{h}}(V, W)$$

(where I'm dropping making explicit the restriction operation).

Verma modules, which we'll use in highest-weight theory, are an example of this. They can be defined as

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a Borel subalgebra, and $\mathbf{C}_\lambda$ is the one-dimensional representation of $\mathfrak{b}$ on which $\mathfrak{n}^+$ acts trivially, $\mathfrak{h}$ with weight λ. Here Frobenius reciprocity tells us that

$$Hom_{\mathfrak{g}}(M_\lambda, V) = Hom_{\mathfrak{b}}(\mathbf{C}_\lambda, V)$$

Note that these representations will be infinite dimensional, and not necessarily irreducible. One of our constructions of the finite dimensional representations of a complex semi-simple Lie algebra $\mathfrak{g}$ will proceed by starting with this induced module, then looking for a finite-dimensional quotient by a sub-module (also infinite-dimensional).

3 Induction for Lie groups

For Lie groups what is easy to work with is an analog of $Hom_{\mathbf{H}}(\mathbf{C}G, V)$. We'll define induction by

Definition 4. *Given a Lie group* G *with subgroup* H*, and* (π, V) *a representation of* H*, the induced representation of* G *is*

$$Ind_H^G(V) = Map_H(G, V)$$

where

$$Map_H(G, V) = \{f : G \to V, \text{ such that } f(gh) = (\pi(h^{-1})f(g)) \;\; \forall\, g \in G, h \in H\}$$

One has Frobenius reciprocity in the form

$$Hom_G(V, Ind_H^G W) = Hom_H(V, W)$$

Depending on the sort of group one is dealing with, note that one additionally has to specify what class of maps one is dealing with. In the first problem set, one exercise will be to prove Frobenius reciprocity in the Lie algebra case, and in the Lie group case, for compact Lie groups with continuous maps.

For non-compact Lie groups and various choices of spaces of maps, one needs to separately check whether Frobenius reciprocity will hold. In the non-compact case, induction won't necessarily take unitary representations to unitary representations, and one may want to change one's definition of induction to a unitary version.

There is an analog of the Hecke algebra in the Lie group case, using the convolution algebra of bi-invariant functions on the group (or bi-invariant differential operators). Cases where this algebra is commutative give "“Gelfand pairs” of groups $H \subset G$.

Note that another language for describing the same space of maps is that of homogeneous vector bundles. For any representation V of H, one can define an associated vector bundle E_V over the homogeneous space G/H, and its space of sections $\Gamma(E_V)$ can be identified with $Map_H(G, V)$, giving another interpretation of the induced representation. For more details of this, see an earlier version of these notes[2].

For the case of the trivial representation of H, the induced representation is just a space $Fun(G/H)$ of functions on G/H and Frobenius reciprocity says that

$$Hom_G(V, Fun(G/H)) = Hom_H(V, \mathbf{C})$$

For compact H, G and V a finite dimensional representation irreducible representation of G, this says that the multiplicityof V in the decomposition of $Fun(G/H)$ into irreducibles will be given by the multiplicity of the trivial representation of H in V.

Some suggested places to find much more detail about this are [3] and [4].

4 An Example: spherical harmonics

A simple example of how this works is behind an analog of Fourier analysis for the sphere, where one decomposes functions on the 2-sphere using orthogonal functions called “spherical harmonics”. These are often written using the notation

$$Y^l_m(\phi, \theta)$$

where ϕ and θ are angles parametrizing the sphere, l is a non-negative integer and m is an integer taking on the $2l+1$ values $-l, -l+1, \cdots, l-1, l$.

This is a crucial example to understand, since it's the simplest non-trivial one for the geometric approach to constructing representations of compact Lie groups. Consider

$$G = SU(2) = Spin(3),\ T = U(1) = Spin(2)$$

and the quotient

$$G/T = SO(3)/SO(2) = S^2$$

The irreducible representations of T are labelled by an integer n and the irreducible representations V_k of $SU(2)$ correspond to non-negative values of k

(dominant weights in this case) where k is the largest integer that occurs when you restrict V_k to be a $U(1)$ representation. These irreducible representations of $SU(2)$ are the same as those for $\mathfrak{sl}(2, \mathbf{C})$, and you should have seen a construction of these last semester as homogeneous polynomials of degree k in two variables. The $U(1)$ weights of V_k are

$$-k, -k+2, \cdots, k-2, k$$

and Frobenius reciprocity tells us that

$$\dim \mathrm{Hom}_{SU(2)}(V_k, \Gamma(L_n)) = \dim \mathrm{Hom}_{U(1)}(V_k, n)$$

Here L_n is the complex line bundle over $SU(2)/U(1)$ constructed by the associated bundle construction using the $U(1)$ representation on $\mathbf{C}$ labeled by n, and

$$Ind_{U(1)}^{SU(2)}(n) = \Gamma(L_n)$$

The right hand side is the multiplicity of the weight n in V_k. We can see that the decomposition of $\Gamma(L_n)$ into irreducibles will include, with multiplicity one, all V_k where k is the same parity as n and $k \geq n$.

Spherical harmonics correspond to the case $n = 0$, in this case L_0 is the trivial $\mathbf{C}$ bundle and $\Gamma(L_0)$ is just the space of complex-valued functions on S^2. All V_k with k even will occur in $\Gamma(L_0)$, each with multiplicity one. The V_k with k odd do not occur since they do not include the weight 0. This gives spherical functions that are not only $SU(2)$ representations but also $SO(3)$ representations, with dimensions $1, 3, 5, \cdots$. V_k, k even corresponds to the space of spherical harmonics $Y_m^l(\phi, \theta)$ with $l = k/2$.

Note that here we are using all sections of the line bundles L_n and getting an infinite-dimensional induced representation that includes an infinite set of irreducibles. To construct a given irreducible, we would like to have a way of picking it out of $\Gamma(L_n)$. You may have seen these line bundles before in an algebraic geometry class (they're powers of the tautological line bundle over $\mathbf{C}P^1$), and in that context it's clear that they are holomorphic objects and one can restrict attention to holomorphic sections. It turns out that when one does this, one gets exactly one irreducible as desired. This is the simplest example of the Borel-Weil theorem, which we will study later in detail.

References

[1] Knapp, A. *Lie Groups Beyond an Introduction, Second Edition* Birkhauser, 2002.

[2] `http://www.math.columbia.edu/ woit/notes13.pdf`

[3] Segal, G., Lie Groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.

[4] Sepanski, M., *Compact Lie Groups,* Springer-Verlag, 2006.