Highest-weight Theory: Borel-Weil

Math G4344, Spring 2012

1 The Borel-Weil theorem

We'll now turn to a geometric construction of irreducible finite-dimensional representations, using induction on group representations, rather than at the Lie algebra level. In this section G will be a compact Lie group, T a maximal torus of G.

Recall that according to the Peter-Weyl theorem

$$L^2(G) = \widehat{\bigoplus_i} V_i \otimes V_i^*$$

with the left regular representation of G on $L^2(G)$ corresponding to the action on the factor V_i and the right regular representation corresponding to the dual action on the factor V_i^*. Under the left regular representation $L^2(G)$ decomposes into irreducibles as a sum over all irreducibles, with each one occuring with multiplicity dim V_i (which is the dimension of V_i^*). To make things simpler later, we'll interchange our labeling of representations and use Peter-Weyl in the form

$$L^2(G) = \widehat{\bigoplus_i} V_i^* \otimes V_i$$

i.e. the left regular representation will be on V_i^*, the right regular on V_i.

Recall also that we can induce from representations $\mathbf{C}_\lambda$ of T to representations of G, with the induced representation interpretable as a space of sections of a line bundle L_λ over G/T.

$$Ind_T^G(\mathbf{C}_\lambda = \Gamma(L_\lambda$$

Here

$$\Gamma(L_\lambda) \subset L^2(G)$$

is the subspace of the left-regular representation picked out by the condition

$$f(gt) = \rho_\lambda(t^{-1})f(g)$$

Here the representation $(\rho_\lambda, \mathbf{C})$of T is one dimensional and if $t = e^H$

$$\rho_\lambda(t^{-1}) = e^{-\lambda(H)}$$

where λ is an integral weight in $\mathbf{t}^*$.

This condition says that under the action of the subgroup $T_R \subset G_R$, $\Gamma(L_\lambda)$ is the subspace of $L^2(G)$ that has weight $-\lambda$. In other words

$$\Gamma(L_\lambda) = \widehat{\bigoplus_i} V_i^* \otimes (V_i)_{-\lambda}$$

or

$$\Gamma(L_{-\lambda}) = \widehat{\bigoplus_i} V_i^* \otimes (V_i)_\lambda$$

where $(V_i)_\lambda$ is the λ weight space of V_i.

Our induced representation $\Gamma(L_{-\lambda})$ thus breaks up into irreducibles as a sum over all irreducibles V_i^*, with multiplicity given by the dimension of the weight λ in V_i. We can get a single irreducible if we impose the condition that λ be a highest weight since by the highest-weight theorem λ will be the highest weight for just one irreducible representation.

So if we impose the condition on $\Gamma(L_{-\lambda})$ that infinitesimal right translation by an element of a positive root space give zero, we will finally have a construction of a single irreducible: it will be the dual of the irreducible with highest weight λ. It turns out (see discussion of complex structures in next section, more detail in [2]) that imposing the condition of infinitesimal right invariance under the action of $\mathfrak{n}^+$ on $\Gamma(L_{-\lambda})$ is exactly the holomorphicity condition corresponding to using the complex structure on G/T to give L_λ the structure of a holomorphic line bundle. So, on the subspace

$$\Gamma_{hol}(L_{-\lambda}) \subset \Gamma(L_{-\lambda}) \subset L^2(G)$$

we have

$$\begin{aligned} \Gamma_{\text{hol}}(L_{-\lambda}) &= \widehat{\bigoplus_i} V_i^* \otimes \{v \in V_i : \begin{cases} \mathfrak{n}^+ v = 0 \\ v \in (V_i)_\lambda \end{cases} \} \\ &= \bigoplus_{V_i \text{ has highest-weight } \lambda} V_i^* \otimes (V_i)_\lambda \\ &= (V^\lambda)^* \otimes \mathbf{C} \end{aligned}$$

where V^λ is the irreducible representation of highest weight λ.

The Borel-Weil version of the highest-weight theorem is thus:

Theorem 1 (Borel-Weil). *As a representation of G, for a dominant weight λ, $\Gamma_{hol}(L_{-\lambda})$ is the dual of a non-zero, irreducible representation of highest weight λ. All irreducible representations of G can be constructed in this way.*

For an outline of the proof from the point of view of complex analysis, see [1] chapter 14. Note however, that we have shown how the Borel-Weil theorem is related to the highest-weight classification of irreducible representations discussed using Lie algebras and Verma modules, with a different explicit construction of the representation. The Lie algebra argument made clear that having a dominant, integral highest weight is a necessary condition for a finite dimensional irreducible. The Verma module construction was such that it was not so easy to see that these conditions were sufficient for finite dimensionality. Finite dimensionality can be proved for the Borel-Weil construction using either

- General theorems of algebraic geometry to show that

$$\Gamma_{hol}(L_\lambda) = H^0(G/T, \mathcal{O}(L_\lambda))$$

is finite dimensional (sheaf cohomology of a holomorphic bundle over a compact projective variety is finite-dimensional), or

- Hodge theory. Picking a metric the Cauchy-Riemann operator has an adjoint, and the corresponding Laplacian is elliptic. An elliptic operator on a compact manifold has finite-dimensional kernel.

2 Flag manifolds and complex structures.

We need to show that the holomorphicity condition on the space of sections $\Gamma(L_\lambda)$ corresponds to imposing the condition that Lie algebra elements in the positive root spaces act trivially, as vector fields corresponding to the infinitesimal right-action of the group. For a detailed argument, see [2], section 7.4.3. The complex geometry involved goes as follows.

We'll need the general notion of what a complex structure on a manifold is. To begin, on real vector spaces:

Definition 1 (Complex structure on a vector space). *Given a real vector space V of dimension n, a complex structure is a non-degenerate operator J such that $J^2 = -\mathbf{1}$. On the complexification $V \otimes \mathbf{C}$ it has eigenvalues i and $-i$, and an eigenspace decomposition*

$$V \otimes \mathbf{C} = V^{1,0} \oplus V^{0,1}$$

$V^{1,0}$ is a complex vector space with complex dimension n, with multiplication by i given by the action of J, $V^{0,1}$ its complex conjugate.

For manifolds

Definition 2 (Almost complex manifold). *A manifold with a smooth choice of a complex structure J_x on each tangent space $T_x(M)$ is called an almost complex manifold.*

and

Definition 3 (Complex manifold). *A complex manifold is an almost complex manifold with an integrable complex structure, i.e. the Lie bracket of vector fields satisfies*

$$[T^{1,0}, T^{1,0}] \subset T^{1,0}$$

By the Newlander-Nirenberg theorem, having an integrable complex structure implies that one can choose complex coordinate charts, with holomorphic transition functions, and thus have a notion of which functions are holomorphic.

Going back to Lie groups and case of the manifold G/T, to even define the positive root space, we need to begin by complexifying the Lie algebra

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and then making a choice of positive roots $R^+ \subset R$

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

Note that the complexified tangent space to G/T at the identity coset is

$$T_{eT}(G/T) \otimes \mathbf{C} = \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and a choice of positive roots gives a choice of complex structure on $T_{eT}(G/T)$, with decomposition

$$T_{eT}(G/T) \otimes \mathbf{C} = T^{1,0}_{eT}(G/T) \oplus T^{0,1}_{eT}(G/T) = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

While $\mathfrak{g}/\mathfrak{t}$ is not a Lie algebra,

$$\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \text{ and } \mathfrak{n}^- = \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

are each Lie algebras, subalgebras of $\mathfrak{g}_{\mathbf{C}}$ since

$$[\mathfrak{n}^+, \mathfrak{n}^+] \subset \mathfrak{n}^+$$

(and similarly for $\mathfrak{n}^-$). This follows from

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The fact that these are subalgebras also implies that the almost complex structure they define on G/T is actually integrable, so G/T is a complex manifold.

Note that the choice of positive roots is not unique or canonical. There are $|W|$ inequivalent choices that will work. The Weyl group acts on the inequivalent complex structures. In particular, it permutes the Weyl chambers, and it is the choice of positive roots that determines which Weyl chamber is the dominant one. Changing choice of positive roots will correspond to changing choice of complex structure. We'll see later on that a representation appearing as a holomorphic section (in H^0) with respect to one complex structure, appears in higher cohomology when one changes the complex structure.

The choice of a decomposition into positive and negative roots takes the original $\mathfrak{g}/\mathfrak{t}$, which is not a Lie subalgebra of $\mathfrak{g}$, and, upon complexification, gives two Lie subalgebras instead:

$$\mathfrak{g}/\mathfrak{t} \otimes \mathbf{C} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

Recall that in the case $\mathfrak{u}(n)$, this corresponds to the fact that, upon complexification to $\mathfrak{gl}(n, \mathbf{C})$, the non-diagonal entries split into two nilpotent subalgebras: the upper and lower triangular matrices.

Another important Lie subalgebra is

$$\mathfrak{b} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{n}^+$$

This is the Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$. One also has parabolic subalgebras, those satisfying

$$\mathfrak{b} \subseteq \mathfrak{p} \subset \mathfrak{g}_{\mathbf{C}}$$

The Borel subalgebra is the minimal parabolic subalgebra. Other parabolic subalgebras can be constructed by adding to the positive roots some of the negative roots, with the possible choices corresponding to the nodes of the Dynkin diagram.

Corresponding to the Lie sub-algebras $\mathfrak{n}^+, \mathfrak{b}, \mathfrak{p}$ one has Lie subgroups N^+, B, P of $G_{\mathbf{C}}$. One can identify

$$G/T = G_{\mathbf{C}}/B$$

and another approach to Borel-Weil theory would be to do "holomorphic induction", inducing from a one-dimensional representation of B on $\mathbf{C}$ to a representation of $G_{\mathbf{C}}$ using holomorphic functions on $G_{\mathbf{C}}$.

One can see that the complex manifolds $G/T = G_{\mathbf{C}}/B$ and $G_{\mathbf{C}}/P$ are actually projective varieties as follows (for more details, see [3]):

Pick a highest-weight vector $v_\lambda \in V^\lambda$ and look at the map

$$g \in G_{\mathbf{C}} \to [gv_\lambda] \in P(V^\lambda)$$

i.e. the orbit in projective space of the line defined by the highest-weight vector. For a generic dominant weight, the Borel subgroup B will act trivially on this line, for weights on the boundary of the dominant Weyl chamber one gets larger stabilizer groups, the parabolic groups P. The orbit can be identified with $G_{\mathbf{C}}/B$ or $G_{\mathbf{C}}/P$, and this gives a projective embedding.

In the special case that the orbit is the full projective space, one can understand the Borel-Weil theorem in the following way:

Given a projective space $P(V)$, one can construct a "tautological" line bundle above it by taking the fiber above a line to be the line itself. In the complex case, this give a holomorphic bundle L, one that has no holomorphic sections. But for each element of V^*, one can restrict this to the line L, getting a section of the dual bundle $\Gamma(L^*)$. It turns out this is an isomorphism

$$V^* = \Gamma_{hol}(L^*)$$

and more generally one has

$$\text{homogeneous polys on V of degree n} = \Gamma_{hol}((L^*)^{\otimes n})$$

i.e. the sections of the n'th power of the dual of the tautological bundle are the homogeneous polynomials on V of degree n. These give special cases of the Borel-Weil theorem, and we'll see explicitly how this works for $V = \mathbf{C}^2$ in the next section.

3 The Borel-Weil theorem: Examples

Recall that for the case of $G = SU(2)$, we had an explicit construction of irreducible representations in terms of homogeneous polynomials in two variables. Such a construction can be interpreted in the Borel-Weil language by identifying holomorphic sections explicitly in terms of homogeneous polynomials. We will begin by working this out for the SU(2) case. For this case

$$G/T = SU(2)/U(1) = SL(2, \mathbf{C})/B = \mathbf{C}P^1$$

the space of complex lines in $\mathbf{C}^2$. Elements of $SL(2, \mathbf{C})$ are of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1$$

and elements of the subgroup B are of the form

$$b = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$$

B can also be defined as the subgroup that stabilizes a standard complex line in $\mathbf{C}^2$, and one can check that for $b \in B$

$$b \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The subgroup N^+ of B in this case is the matrices of the form

$$n = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

and the subgroup $T_{\mathbf{C}}$ is elements of the form

$$t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

The space of holomorphic sections $\Gamma_{\mathrm{hol}}(L_k)$ will be functions on $SL(2, \mathbf{C})$ such the subgroup N acts trivially from the right and the subgroup $T_{\mathbf{C}}$ acts via a character of T, which corresponds to an integer k. More explicitly

$$\Gamma_{\mathrm{hol}}(L_{-k}) = \{f : SL(2, \mathbf{C}) \to \mathbf{C}, f(gb) = \alpha^k f(g) \ \forall \ b \in B\}$$

We'll analyze what this equivariance condition says in two parts. First choosing $b \in N$, since

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta'\alpha + \beta \\ \gamma & \beta'\gamma + \delta \end{pmatrix}$$

the condition $f(gb) = f(g)$ means that f depends only on the first column of the matrix.

Secondly, choosing $b \in T_{\mathbf{C}}$,

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ 0 & (\alpha')^{-1} \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & (\alpha')^{-1}\beta \\ \gamma\alpha' & (\alpha')^{-1}\delta \end{pmatrix}$$

so the equivariance condition $f(gb) = (\alpha')^k f(g)$ implies that

$$f(\alpha' \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}) = (\alpha')^k f(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix})$$

so our homogeneous polynomials of degree k in two variables (α, γ) provide holomorphic

sections in $\Gamma_{\mathrm{hol}}(L_k)$ and it turns out these are all such sections.

Another way of thinking about how to produce the appropriate holomorphic function on $SL(2, \mathbf{C})$ out of a homogeneous polynomial $P(z_1, z_2)$ is by the map

$$P \rightarrow f(g) = P(g \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = P(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = P(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix})$$

In the more general case of $G = SU(n)$, representations on polynomials in n variables correspond to sections of a line bundle over

$$SU(n)/U(n-1) = SL(n, \mathbf{C})/P = \mathbf{CP}^{n-1}$$

the space of complex lines in $\mathbf{C}^n$. The parabolic subgroup P in this case can be taken to be the set of all matrices with zero in the first column, except for the diagonal element in the first row. The equivariance condition defining line bundles over this space is the same as in the $SU(2)$ case and the relation of holomorphic sections and homogeneous polynomials is much the same.

Note that for $G = U(n)$, $G_{\mathbf{C}}/B = Fl(n)$, the space of flags in $\mathbf{C}$ has an obvious map to any of the partial flag manifolds $G_{\mathbf{C}}/P$ such as $G_{\mathbf{C}}/P = \mathbf{CP}^{n-1}$, given by just forgetting some parts of the flag. In the case of $G_{\mathbf{C}}/P = \mathbf{CP}^{n-1}$, the map just forgets all parts of the flag except for the complex line. The line bundle on $G_{\mathbf{C}}/B$ of the Borel-Weil theorem is just the pull-back under this forgetting map of the one we constructed on $\mathbf{CP}^{n-1}$ with homogeneous polynomials as its holomorphic sections.

The fundamental representations of $SU(n)$ include the $k = 1$ case above which is just the defining representation on $\mathbf{C}^n$, but also include the representations on the higher degree parts of $\Lambda^*(\mathbf{C}^n)$. The representation on $\Lambda^k(\mathbf{C}^n)$ corresponds to holomorphic sections of a certain line bundle over the Grassmanian

$$Gr(k, n) = SL(n, \mathbf{C})/P = \frac{U(n)}{U(k) \times U(n-k)}$$

For more details about this, the Borel-Weil theorem and its relation to the examples discussed here, see Chapter 11 and 14 of [1].

References

[1] Segal, G. Lectures on Lie groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.

[2] Sepanski, M. *Compact Lie Groups,* Springer-Verlag, 2007. Chapter 7.4

[3] Segal, G. *Loop Groups* Oxford University Press, 1985. Chapter 2.