MODERN GEOMETRY FINAL: DEC. 15, 2005
DUE: WEDNESDAY, DEC. 22, 2005 at 5:00 PM
Return to Shavonn in the Math Office by the deadline

This exam is take home and open notes. You may consult your class notes, but NOT any other sources (books, journal articles, etc.) You may not discuss the exam with anyone else except me.

1) (a) Let M and N be smooth manifolds and let I be the unit interval. Let $F\colon M \times I \to N$ be a smooth map and denote by $F_0\colon M \to N$ and $F_1\colon M \to N$ the restrictions of F to $M \times \{0\}$ and $M \times \{1\}$, respectively. Show that if ω is a closed k-form on N, then there is a $(k-1)$ form η on M satisfying the equation

$$F_1^*\omega - F_0^*\omega = d\eta.$$

Conclude that $F_1^* = F_0^*$ as maps $H^*_{dR}(N) \to H^*_{dR}(M)$.

(b) For any smooth manifold M and any open interval J containing the origin show that the inclusion $M \times \{0\} \to M \times J$ induces an isomorphism on deRham cohomology.

(c) Let $U \subset \mathbb{R}^n$ be a star-like open subset of $\mathbb{R}^n$. (This means that for every $x \in U$ the straight-line segment connecting x to the origin in $\mathbb{R}^n$ is contained in U.) Show that $H^k_{dR}(U) = 0$ for all $k > 0$.

2) Fix $n > 1$; let S^n be the unit n sphere and $S^{n-1} \subset S^n$ the equatorial sphere of codimension one. Let $A \subset S^n$ be an open annular neighborhood of the equator $S^{n-1} \subset S^n$. Let $A' \subset A$ be an other annular neighborhood of the equator with the property that the closure of A' is contained in A. Let $H_+ \subset S^n$ be the union of the upper hemisphere and A and H_- the union of the lower hemisphere and A. Similarly, define $H'_\pm$ using A' instead of A. Let ω be a closed k-form on S^n for some $k > 0$. Show that there are forms $\eta_\pm$ on $H_\pm$ satisfying

$$d\eta_\pm = \omega|_{H_\pm}.$$

Show that $\mu = (\eta_+|_A - \eta_-|_A)$ is a closed form. Show that the class $[\mu] \in H^{k-1}_{dR}(S^{n-1})$ is independent of the choices. Use this to define a map $H^k_{dR}(S^n) \to H^{k-1}_{dR}(S^{n-1})$ and show that it is independent of the choice of A. Show that this map is an isomorphism. [Hint: Use A' and A together for this.] Compute the deRham cohomology of S^n for every $n \geq 1$.

3) (a) Let M and N be smooth manifolds and let $\varphi\colon |K^k| \to M$ and $\psi\colon |L^\ell| \to N$ smooth cycles. For any differential forms $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(N)$ define a form $\omega \otimes \eta \in \Omega^{k+\ell}(M \times N)$. Show that

$$\int_{\varphi\times\psi(|K\times L|)} \omega \otimes \eta = \left(\int_{\varphi(|K|)} \omega\right) \cdot \left(\int_{\psi(|L|)} \eta\right).$$

(b) Now suppose that $M = N$ and let $\Delta\colon M \to M \times M$ be the diagonal embedding ($\Delta(m) = (m, m)$ for all $m \in M$). Show that

$$\Delta^* \omega \otimes \eta = \omega \wedge \eta.$$

(c) For this part you may assume that for any class $\alpha \in H_r(M)$ in integral singular homology, the image $\Delta_*(\alpha)$ can be written as a sum of products of classes $\sum_i a_i \times b_i$ where $a_i \in H_{k(i)}(M)$ and $b_i \in H_{r-k(i)}(M)$ are integral singular classes in the first and second factors respectively. Using this show that if ω and η are closed forms with integral periods on all (integral singular) homology classes of M then $\omega \wedge \eta$ is also a closed differential form with integral periods on every (integral singular) homology class.

4) Let M be a smooth manifold and $\gamma\colon (-\epsilon, \epsilon) \to M$ a smooth curve with $\gamma(0) = p$. Show that $\gamma'(0)$ defines a tangent vector in T_pM, the tangent space to M at p. Let V be a finite dimensional real vector space and denote by $P(V)$ the projective space of lines in V. Fix a line $L \subset V$ and let $\varphi\colon L \to V/L$ be a homomorphism. For any $v \in L \setminus \{0\}$ let $w = \varphi(v) \in V/L$, and let $\tilde{w} \in V$ be a lifting of w. Show that for t sufficiently small, $v + t\tilde{w}$ defines a point $[v + t\tilde{w}]$ in $P(V)$, and define a path $\gamma(t) = [v + t\tilde{w}]$. Show that this path is smooth. Show that $\gamma'(0) \in T_{[v]}P(V)$ depends only on φ and not on the choices of v, w and $\tilde{w}$. Show that this association defines a linear isomorphism $\mathrm{Hom}(L, V/L) \to T_LP(V)$. Show that this identification is natural for linear maps between finite dimensional real vector spaces.

5) (a) Let $\mathbf{H}$ be the upper half-plane

$$\mathbf{H} = \{z \in \mathbb{C} \big| \mathrm{Im}(z) > 0\}.$$

We identify $T_z\mathbf{H}$ with $\mathbb{C}$ under the embedding $\mathbf{H} \subset \mathbb{C}$. Now define a riemannian metric on $\mathbf{H}$ as follows: suppose that z_1, z_2 are tangent vectors at $z \in \mathbf{H}$ then

$$\langle z_1, z_2 \rangle = \frac{\mathrm{Re}(z_1 \overline{z_2})}{(\mathrm{Im} z)^2}.$$

Using real and imaginary parts x and y this metric is also written

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Show that this is indeed a riemannian metric on $\mathbf{H}$. Compute the Christoffel symbols of the metric (in terms of x and y) and write down the geodesic equation for this riemannian manifold.

(b) Show that vertical lines are images of geodesics. Determine a parameterization that makes a given vertical line a geodesic.

(c) Let the group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| a, b, c, d \text{ are real and } ad - bc = 1 \right\}$$

acts by linear fractional transformations on $\mathbf{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

Show that this is a smooth action and the subgroup $\pm\mathrm{Id}$ is the largest subgroup that acts trivially. Setting $PSL(2,\mathbb{R})$ equal to the quotient of $SL(2,\mathbb{R})$ by $\{\pm\mathrm{Id}\}$ that the $PSL(2,\mathbb{R})$-action preserves the metric defined above, so that it acts as a group of isometries.

(d) Show that $PSL(2,\mathbb{R})$ acts transitively on $\mathbf{H}$ and that the subgroup fixing any point $p \in \mathbf{H}$ is isomorphic to the circle group.

(e) Show that $PSL(2,\mathbb{R})$ is the full group of orientation-preserving isometries. [Hint: Show that an isometries of $\mathbf{H}$ that fixes a point and whose differential at that point is the identity is in fact the identity map.]

(f) Find the possible images of vertical lines by the action of elements in $PSL(2,\mathbb{R})$. Show that any such image is a geodesic. Describe all geodesics in $\mathbf{H}$.

6) Find all elements of finite order in $SL(2,\mathbb{Z})$. [Hint: Consider the traces of these elements.] Show that an element of $PSL(2,\mathbb{Z})$ acts on $\mathbf{H}$ with a fixed point if and only if it is of finite order. Consider the subgroup $\Gamma(2)$ consisting of all the matrices in $SL(2,\mathbb{Z})$ that are congruent modulo two to the identity. Show that the only elements in $\Gamma(2)$ of finite order are $\pm\mathrm{Id}$. Show that $\Gamma(2)/\{\pm\mathrm{Id}\}$ acts freely and properly discontinuously on $\mathbf{H}$. Describe the quotient surface of this action.

7) Suppose that M^n is a closed, oriented n-dimensional manifold. Show that $H^n_{dR}(M^n)$ is non-trivial.

8)Suppose that $\alpha \in H^1_{dR}(M)$. Show that integrating a closed form representative for α over loops γ in M defines a homomorphism $\pi_1(M) \to \mathbb{R}$. Show that this homomorphism is trivial if and only if $\alpha = 0$ in $H^1_{dR}(M)$.

9) Recall that a manifold M is symplectic if it possess a closed two-form ω with the property that ω is non-degenerate at each point of M, non-degenerate in the sense that the contraction mapping

$$T_pM \to T^*_pM$$

defined by $\tau \mapsto \omega(\tau,\cdot)$ is an isomorphism at every $p \in M$. Show that any symplectic manifold is orientable. Show that any closed symplectic manifold has the property the $H^{2k}(M) \neq 0$ for all $2k \leq \dim(M)$.

10) A Lie group is a smooth manifold G such that multiplication $G \times G \to G$, $(g,h) \mapsto gh$, and the inverse mapping $G \mapsto G$, $g \mapsto g^{-1}$, are smooth. The Lie algebra of G is the vector space of all left-invariant vector fields. That is to say all vector fields χ on G with the property that $D(g\cdot)\chi(h) = \chi(gh)$, where $g\cdot$ is left multiplication by g and $D(g\cdot)$ is the differential of this smooth mapping. Show that the vector space of left-invariant vector fields is finite dimensional and in fact is identified with the tangent space to G at the identity element. Show that the Lie

bracket of two left-invariant vector fields is itself left-invariant, so that these vector fields form a finite dimensional Lie algebra. It is called the Lie algebra of the Lie group. Give a basis (generators and relations) for the Lie algebra of $SO(3)$.