Computability of a Topological Poset

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Abstract

For subspaces X and Y of $\mathbb{Q}$ the notation $X \leq_h Y$ means that X is homeomorphic to a a subspace of Y and $X \sim Y$ means $X \leq_h Y \leq_h X$. The resulting set $\mathcal{P}(\mathbb{Q})/\sim$ of equivalence classes $\bar{X} = \{Y \subseteq \mathbb{Q} : Y \sim X\}$ is partially-ordered by the relation $\bar{X} \leq_h \bar{Y}$ if $X \leq_h Y$. In a previous paper by the author it was established that this poset is essentially determined by considering only the scattered $X \subseteq \mathbb{Q}$ of finite Cantor-Bendixson rank. Results from that paper are extended to show that this poset is computable.

Key words: Embedding, partially-ordered set, scattered, metric space, computable
1991 MSC: 54H10, 54H12, 54D35, 06A06

1 Introduction

This paper is mainly a continuation of [1]. The notation and terminology from that paper are used here. All topological spaces considered are assumed to be scattered countable metric spaces X with finite (Cantor-Bendixson) rank (denoted $N(X)$). We may assume each of these is a subspace of $\mathbb{Q}$. Here we show that the partially-ordered set $\mathbb{A} = \{X \subseteq \mathbb{Q} : 1 < N(X) < \omega\}/\sim$ (ordered be embeddability) originally defined in [1 Section 4] can be generated by a computer program. En route to this conclusion we give a classification (or a *presentation* perhaps) of homeomorphism types of countable metric spaces with finite Cantor-Bendixson rank. For example, in Section 3, we give a complete list of homeomorphism types of spaces of rank ≤ 4, and a presentation of the embeddability ordering of these spaces. Our focus here is on a closer study of the *types* introduced in Definition 20 of the aforementioned paper. There we defined the type of an isolated point to be $(\emptyset, \emptyset)$ and then inductively defined the type of a point $p \in X$ with higher rank to be (A, B) where A is the set of

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types τ so that every neighborhood of p in X contains infinitely many points of type τ and B is the set of types σ so that every neighborhood of p in X contains a closed infinite set of points of type σ.

2 The Combinatorics of Types

In [1 Definition 21] we defined T_n to be the set of types arising from points in spaces X with $N(X) < n$. It is easy to see that T_n is finite for each $n \in \mathbb{N}$ so we would like to determine $|T_n|$ as a function of n. The results presented here will reduce that question to a (difficult) finite combinatorial problem, which we do not solve. We only give a recursive algorithm (Theorem 1) for computing T_{n+1} from T_n, but we have no closed form expression for $|T_n|$. Another difficulty concerns the functions $f_X : T_{N(X)+1} \to \omega + 1 + 1$ defined in [1 Definition 23]. Recall that $f_X(\tau)$ was defined to be the number of points of type τ in X with the convention that if there was a *closed* infinite set, we counted its cardinality as "$\omega + 1$". Which functions $f : T_n \to \omega + 1 + 1$ are actually of the form $f = f_X$ for a space X? We answer this in Theorem 3. Also note that [1 Lemma 27] is quite weak because there are certainly scattered spaces $X, Y \subseteq \mathbb{Q}$ where X embeds in Y even though $f_X(\tau) > f_Y(\tau)$ for certain types τ (e.g. take $X := M_0, Y := M_1, \tau = (\{(\emptyset, \emptyset)\}, \emptyset))$. This weakness is remedied below. Many natural questions are left unanswered. For example, we have no general formula for the size of a maximal anti-chain in T_n as a function of n.

Note that for each $\tau \in T_n$ there is a way of associating a canonical scattered space $S(\tau) \subseteq \mathbb{Q}$ to the type τ. Since $\tau = (A, B)$ arises as the type of a point p in some finite rank space $X \subseteq \mathbb{Q}$, it is always possible to choose a neighborhood U of p in X containing points of type σ only for $\sigma \in A$ and containing closed infinite sets of points of type σ only for $\sigma \in B$. Let us call such an open-and-closed neighborhood *standard*. Now take $S(\tau) := U$. The space $S(\tau)$ is the unique space (by [1 Theorem 24]) where $f_{S(\tau)}(\sigma)$ is $\omega + 1$ when $\sigma \in B$, ω when $\sigma \in A \setminus B$, 1 when $\sigma = \tau$ and 0 otherwise.

Theorem 1 *$T_0 = T_1 = \emptyset, T_2 = \{(\emptyset, \emptyset)\}$ and for each finite $n \geq 2$, $T_{n+1} \setminus T_n$ consists of all elements $(A, B) \in \mathcal{P}(T_n) \times \mathcal{P}(T_n)$ satisfying all of the following:*

(i) $A \cap (T_n \setminus T_{n-1}) \neq \emptyset$
(ii) $B \subseteq A$
(iii) $(A', B') \in A \Rightarrow A' \subseteq A$ and $B' \subseteq B$
(iv) $(A', B') \in B \Rightarrow A' \subseteq B$ and $B' \subseteq B$

PROOF. We show that this recursive definition is correct using induction on n. Suppose that the recursive definition of $T_1, T_2, \ldots, T_n$ given above is correct (certainly it is up to $n = 1$). Now assume $n \geq 2$ and note that the set $T_{n+1} \setminus T_n$ is the set of types of $(n-1)$-isolated points. If $(A, B) \in T_{n+1} \setminus T_n$ then (A, B) arose as the type of some $(n-1)$-isolated point p in a scattered space $X \subseteq \mathbb{Q}$ and hence it satisfies (ii),(iii), and (iv) by [1 Theorem 22] and (i) is simply the observation that there must be a set of $(n-2)$-isolated points converging to p, infinitely many of which must have the same type by [1 Theorem 22(i)]. Now fix some $(A, B) \in \mathcal{P}(T_n) \times \mathcal{P}(T_n)$ satisfying the four conditions and we will show that it actually arises as the type of an $(n-1)$-isolated point in a scattered space $X \subseteq \mathbb{Q}$.

Let $A := \{\tau_1, \ldots, \tau_m, \tau_{m+1}, \ldots, \tau_l\}$ where (by (ii)) we may assume the elements of A are listed so that $A \setminus B = \{\tau_1, \ldots, \tau_m\}$ and $B = \{\tau_{m+1}, \ldots, \tau_l\}$. For $1 \leq j \leq m$, embed a copy of $S(\tau_j)$ into each interval I_n (as described in [1 Section 1] where $n = j \pmod{l}$. For $m+1 \leq j \leq l$, embed a copy of $\oplus_\omega S(\tau_j)$ into each interval I_n where $n = j \pmod{l}$. Add the point 1 to obtain a scattered subspace X of $\mathbb{Q}$.

Note that (by (i)) since some type of an $(n-2)$-isolated point was in A, it follows that 1 is $(n-1)$-isolated in X so $\tau(1; X) = (A^*, B^*) \in T_{n+1} \setminus T_n$. Now we verify that $(A^*, B^*) = (A, B)$. The inclusions $A \subseteq A^*, B \subseteq B^*$ are immediate. Now take some $\sigma \in A^*$. It must be that σ is in the support of $f_{S(\tau_j)}$ for some j with $1 \leq j \leq l$. However, for that to happen, either $\sigma = \tau_j$ (in which case σ must have been in A or B hence $\sigma \in A$ by condition (ii)) or $\sigma \in A'$ where $\tau_j = (A', B')$ (in which case $\sigma \in A$ by conditions (ii),(iii), and (iv)). This shows that $A^* = A$. Now consider a type $\sigma \in B^*$. One (or both) of the following must occur:

Case 1: σ is in the support of $f_{S(\tau_j)}$ for some j with $m+1 \leq j \leq l$. Then if $\sigma = \tau_j$, $\sigma \in B$ is immediate. Otherwise it must be that $\sigma \in A'$ where $\tau_j = (A', B')$ and hence $\sigma \in B$ by condition (iv).

Case 2: $f_{S(\tau_j)}(\sigma) = \omega + 1$ for some j with $1 \leq j \leq m$. That is to say that $\sigma \in B'$ where $\tau_j = (A', B')$ and hence $\sigma \in B$ by conditions (ii) and (iii).

The finite sets T_n can be ordered in a natural way. Declare $\tau \leq_h \sigma$ if and only if $S(\tau) \leq_h S(\sigma)$. This makes $(T_n, \leq_h)$ a finite pre-ordered set. Declaring $\tau \sim \sigma$ if and only if $\sigma \leq_h \tau \leq_h \sigma$ and ordering the resulting equivalence classes in the obvious way makes $(T_n/\sim, \leq_h)$ into a finite partially-ordered set. The next lemma shows that the partially-ordered sets $(T_n/\sim, \leq_h)$ can be constructed via a recursive algorithm. Note that $(T_0/\sim, \leq_h)$ and $(T_1/\sim, \leq_h)$ are empty while $(T_2/\sim, \leq_h)$ is the partially-ordered set with one element.

Lemma 2 *Let $(A, B) \in T_{n+1} \setminus T_n$ and let $(A', B') \in T_{n+1}$. Then $(A', B') \leq_h (A, B)$ if and only if either (i) or (ii) (as below) holds.*

(i) For each $\sigma \in A'$ there is $\tau \in A$ such that $\sigma \leq_h \tau$ and for each $\sigma \in B'$ either
  (1) there is $\tau \in B$ such that $\sigma \leq_h \tau$ or
  (2) there is $(A'', B'') \in A$ and $\tau \in A''$ such that $\sigma \leq_h \tau$
(ii) $(A', B') \in T_n$ and $(A', B') \leq_h \sigma$ for some $\sigma \in A$

PROOF. Clearly condition (ii) is sufficient and it is not too hard to see that condition (i) is sufficient using the "canonical" constructions of $S((A, B))$ and $S((A', B'))$ as in the proof of Theorem 1. To show that either (i) or (ii) is necessary, again use the canonical realizations of $S((A, B))$ and $S((A', B'))$ and suppose there is an embedding $f : S((A', B')) \to S((A, B))$. The point $1 \in S((A', B'))$ is the unique point in that space with type (A', B') and similarly for $1 \in S((A, B))$.

If $f(1) = 1$, argue that condition (i) must be satisfied. Indeed, for each $\sigma \in A'$, $S(A', B')$ contains infinitely many points of type σ, so there must be some type $\tau \in A$ such that infinitely many points of type σ are taken by f to points of type τ. Every neighborhood of a point contains a standard neighborhood of that point, so since f is an embedding, f must take a standard neighborhood of a point of type σ into a standard neighborhood of a point of type τ—this requires $\sigma \leq_h \tau$. For each $\sigma \in B'$, $S(A', B')$ contains a closed infinite set of points each of type σ. Thus some closed infinite set of points all of type σ must be taken by f to an infinite set of points, all of type τ, which is closed in $f[S((A', B'))]$. We claim that τ must be as in (1) or (2). Indeed, if this were not the case, then since (2) fails, for each for each type $\tau' \in A$, $S(\tau')$ contains no points of type τ if $\tau \neq \tau'$ and only one point of type τ when $\tau = \tau'$. Then since (1) fails, for each interval I_n, $I_n \cap S((A, B))$ contains at most one point of type τ. Now since $f(1) = 1 \in f[S((A', B'))]$, it follows that no infinite set of points all of type τ could possibly be closed in $f[S((A', B'))]$.

If $f(1) \neq 1$, then (A', B') cannot be in T_{n+1}. The point $f(1) \in S((A, B))$ has type σ for some $\sigma \in A$ and since f is an embedding, f must take some neighborhood of 1 in $S(A', B')$ into a standard neighborhood of $f(1)$ in $S((A, B))$. Now (ii) is immediate since any neighborhood of 1 in $S((A', B'))$ is homeomorphic to $S((A', B'))$.

The next result classifies the functions $f : T_n \to \omega + 1 + 1$ that arise as f_X for a scattered space $X \subseteq \mathbb{Q}$ with finite rank.

Theorem 3 *A function $f : T_n \to \omega + 1 + 1$ is equal to f_X for a space $X \subseteq \mathbb{Q}$ of finite rank if and only if f satisfies all the following conditions for all $(A, B) \in T_n$.*

(i) $f((A, B)) = \omega + 1 \Rightarrow f(\sigma) = \omega + 1$ for all $\sigma \in A$
(ii) $f((A, B)) > 0 \Rightarrow f(\sigma) \geq \omega$ for all $\sigma \in A$ and $f(\sigma) = \omega + 1$ for all $\sigma \in B$

(iii) $f((A,B)) = \omega \Rightarrow$ *there is* (A', B') *in the support of* f *with* $(A,B) \in A'$

PROOF. To see that (i) is necessary, take a closed infinite set $\{x_1, x_2, \ldots\}$ of points all of type (A,B) in X and fix some $\sigma \in A$. For each n, choose a point a_n of type σ with $d(x_n, a_n) < 1/n$ and argue that $\{a_1, a_2, \ldots\}$ is closed in X. Necessity of (ii) is obvious. To see that (iii) is necessary, take an infinite set of points in X each of type (A,B) (which is not closed in X since $f((A,B)) \neq \omega+1$) and look at the type of any point in the closure of this set.

Now consider a function f satisfying (i),(ii), and (iii). Let $\tau_1, \ldots, \tau_k$ be the types that f takes to $\omega+1$ and let $(\tau_{k+1}, n_1), \ldots, (\tau_{k+m}, n_m)$ be the pairs with $0 < f(\tau_{k+i}) = n_i < \omega$. Let $\tau_i = (A_i, B_i)$ for $i = 1, \ldots, k+m$. Let X be the space $\oplus_{i=1}^{k}(\oplus_\omega S(\tau_i)) \oplus \oplus_{j=1}^{m}(\oplus_{n_j} S(\tau_{k+j})$. We claim $f = f_X$. The types taken by f_X to $\omega+1$ are

(1) $\tau_1 = (A_1, B_1), \ldots, \tau_k = (A_k, B_k)$
(2) each type $\sigma \in A_i$ for some $i = 1, \ldots, k$ and
(3) each type $\sigma \in B_i$ for some $i = 1, \ldots, k+m$.

By definition, f takes the types in (1) to $\omega+1$. Since f satisfies (i), it also takes the types in (2) to $\omega+1$ and since f satisfies (the second part of) (ii), it also takes the types in (3) to $\omega+1$. Thus $f(\tau) = \omega+1$ if and only if $f_X(\tau) = \omega+1$ for all $\tau \in T_\omega$. By construction it is also immediate that f and f_X agree on all the types either takes to a finite ordinal. Suppose $f_X(\sigma) = \omega$. Then $\sigma \neq \tau_i$ for $i = 1, \ldots, k+m$, and $\sigma \in A_i$ for some $i \in \{k+1, \ldots, k+m\}$. Thus $f(\sigma) \geq \omega$ by (the first part of) (ii) and $f(\sigma) \neq \omega+1$ because σ is not in $\{\tau_1, \ldots, \tau_k\}$. So $f(\sigma) = \omega$.

Now suppose $f(\sigma) = \omega$. By (iii) there is a type (C_1, D_1) with $\sigma \in C_1$ and $f((C_1, D_1)) > 0$. Notice that $f((C_1, D_1)) < \omega+1$ by (i). If $f((C_1, D_1)) = \omega$, invoke (iii) and (i) again to find (C_2, D_2) with $(C_1, D_1) \in C_2$ and $0 < f((C_2, D_2)) < \omega+1$. Keep doing this until arriving at a type (C_t, D_t) with $(C_{t-1}, D_{t-1}) \in C_t$ and $0 < f((C_t, D_t)) < \omega$. Then $(C_t, D_t) = (A_{k+i}, B_{k+i})$ for some $i \in \{1, \ldots, m\}$ and by [1 Theorem 22(iii)] we have $A_{k+i} = C_t \supseteq C_{t-1} \supseteq \ldots \supseteq C_1$ hence $\sigma \in A_{k+i}$ and it follows that $f_X(\sigma) = \omega$.

Corollary 4 *Every countable metric space with finite rank is homeomorphic to a space of the form* $\oplus_{i=1}^{m}(\oplus_{\alpha_i} S(\tau_i))$ *for* $\tau_1, \ldots, \tau_m \in T_n$ *and* $\alpha_1, \ldots, \alpha_m \in (\omega+1)$.

The representation above is generally not unique, but it will be (up to reordering) if m is chosen to be minimal. Our final goal will be to give necessary and sufficient conditions on f_X and f_Y for X to embed in Y (when X and Y have finite rank). Before stating the next theorem, we need the following notation:

$S_X := \{\tau \in T_\omega : f_X(\tau) \neq 0\}$ (The support of f_X)
$I_X := \{\tau \in T_\omega : f_X(\tau) = \omega + 1\}$
$F_X := \{\tau \in T_\omega : 0 < f_X(\tau) < \omega\}$
$G_X := \{\tau \in T_\omega : f_X(\tau) = \omega \wedge \exists (A, B), (A', B') \in S_X$ with $\tau \in A \wedge (A, B) \in A'\}$
$C_X := \{\tau \in T_\omega : f_X(\tau) = \omega \wedge \tau$ is not in $G_X\}$

Theorem 5 *Let X and Y be countable metric spaces with finite rank. Then $X \leq_h Y$ if and only if there are functions $\Phi : I_X \to S_Y \setminus F_Y$ and $\Psi : F_X \to \omega^{S_Y}$ satisfying the following five conditions.*

(i) $\tau \leq_h \Phi(\tau)$ for all $\tau \in I_X$
(ii) For all $\tau \in F_X$ and for all $\sigma \in S_X$, $\Psi(\tau)(\sigma) > 0$ implies $\tau \leq_h \sigma$
(iii) $\sum_{\sigma \in S_Y} \Psi(\tau)(\sigma) = f_X(\tau)$ for all $\tau \in F_X$
(iv) $\sum_{\tau \in F_X} \Psi(\tau)(\sigma) \leq f_Y(\sigma)$ for all $\sigma \in S_Y$
(v) For all $\sigma' \in \Phi[I_X] \cap C_Y$ there is some $(A, B) \in S_Y$ with $\sigma' \in A$ such that the inequality in (iv) is strict when $\sigma = (A, B)$.

PROOF. To prove these conditions are necessary, assume we have an embedding $f : X \to Y$. For each τ in I_X, X contains a (there may be many such—choose one) closed infinite set of points of type τ all mapped by f to points of the same type $\Phi(\tau)$ in Y. For each $\sigma \in F_Y$, Y only has finitely many points of type σ, so $\Phi(\tau)$ cannot be in F_Y. For each $\tau \in F_X$ and each $\sigma \in S_Y$, let $\Psi(\tau)(\sigma)$ be the number of points of type τ taken by f to points of type σ. Since f is an embedding, for each point $x \in X$ there is a standard neighborhood of x taken by f into a standard neighborhood of $f(x)$. Thus Ψ and Φ as defined above must satisfy conditions (i) and (ii). Clearly (iii) is satisfied since f is injective and $f_X(\tau)$ is the number of points of type τ in X and each such point is taken by f to a point in Y with *some* type. Condition (iv) is also trivially verified. Let us now verify the most difficult condition: number (v). Fix some $\sigma' \in \Phi[I_X] \cap C_Y$, so there is a type τ in I_X and a closed infinite set C of points of X all with type τ that are taken by f to points of type σ'. Look closely at the way C_Y is defined to see that any infinite set of points of type σ' must have a cluster point (choose one and call it p) whose type (call it (A, B)) is in F_Y. Then $\sigma' \in A$ and the inequality in (iv) must be strict because if not, then every point of type (A, B) in Y would be in $f[X]$. However, this cannot be the case because p cannot be in $f[X]$ because f is an embedding so $f[C]$ should be closed in $f[X]$ and p is in the closure of $f[C]$.

Now we prove that these conditions are sufficient. Assume we are given X, Y and Ψ, Φ satisfying all the conditions. As in the proof of the previous theorem, X can be written in the form $\oplus_{i=1}^{k}(\oplus_\omega S(\tau_i)) \oplus \oplus_{j=1}^{m}(\oplus_{n_j} S(\tau_{k+j})$ where $I_X = \{\tau_1, \ldots, \tau_k\}$ and $F_X := \{\tau_{k+1}, \ldots, \tau_{k+m}\}$. There are only finitely many points in Y of type σ for each type $\sigma = (A, B) \in F_Y$, so choose standard neighborhoods of each such point which are pairwise-disjoint. By possibly making these neighborhoods a bit smaller, we may assume that the complement of the union of all of these neighborhoods contains a closed infinite set of points of type σ'' for each $\sigma'' \in B$ for each $\sigma \in F_Y$ and at least one point of type $\sigma'' \in A$ for each $\sigma \in F_Y$.

For each $\sigma \in S_Y$ and each $j \in \{1, \dots, m\}$, embed $\Psi(\tau_{k+j})(\sigma)$-many of the $S(\tau_{k+j})$ summands into pairwise-disjoint standard neighborhoods of points of type σ (using the previously-chosen neighborhoods when $\sigma \in F_Y$ and ensuring disjointness from all the previously defined neighborhoods when σ is not in F_Y) in Y. Furthermore, by making the standard neighborhoods small enough, we may assume that the complement of the (closure of the) range of the union of these embeddings (call this map f') still contains a closed infinite set of points of type σ'' for each $\sigma'' \in B$ for each $\sigma \in F_Y$ and at least one point of type $\sigma'' \in A$ for each $\sigma \in F_Y$. All of this can be done since conditions (ii) and (iii) are satisfied.

For each $j \in \{1, \dots, k\}$, consider the points of type $\Phi(\tau_j)$ in X. If $\Phi(\tau_j)$ is not in C_Y, then there are infinitely many points of type $\Phi(\tau_j)$ in the complement of the closure of the range of f'. Choose an infinite set of such points with at most one cluster point (not in the closure of the range of the union of the previously-defined embeddings) and inductively choose pairwise-disjoint standard neighborhoods of these points, and use condition (i) to find an embedding of the $\oplus_\omega S(\tau_j)$ summand into an infinite and co-infinite union of these neighborhoods. This method ensures that the same process can be conducted again for any $j' \in \{1, \dots, k\}$ with $\Phi(\tau_j)$ not in C_Y. If $\Phi(\tau_j) \in C_Y$, use condition (v) to find a point p in Y of type (A, B) with $\Phi(\tau_j) \in A$ that is not in the closure of the range of any of the previously-defined embeddings. Then there is an infinite sequence of points all of type $\Phi(\tau_j)$ converging to p (and to no other point). Inductively choose pairwise-disjoint standard neighborhoods of these points and embed (using (i)) the $\oplus_\omega S(\tau_j)$ summand into an infinite and co-infinite union of these neighborhoods. Taking the union of all the embeddings defined in this manner gives an embedding $f : X \to Y$.

Notice that since condition (iii) must be satisfied, to determine whether there are functions Φ and Ψ satisfying conditions (i)-(v), it is not necessary to consider all functions $\Psi : F_X \to \omega^{S_Y}$. It is enough to consider only those Ψ with $\Psi(\tau)(\sigma) \leq \max \{f_X(\tau') : \tau' \in F_X\}$ for all $\sigma \in S_Y$ for all $\tau \in F_X$. Thus only finitely many possibilities for Ψ and Φ need to be considered, and checking whether conditions (i)-(v) are satisfied can be carried out algorithmically in a finite amount of time using Theorem 1 and Lemma 2. Using these results and Theorem 5 proves that $\mathbb{A}$ is a *computable* partially-ordered set in the following sense.

Theorem 6 *There is a decidable subset D of $2^{<\omega}$ (:= the set of all finite* *length binary sequences) and a computable function* $f : 2^{<\omega} \times 2^{<\omega} \to \{0,1\}$ *such that* $(D, \{(d_1, d_2) \in 2^{<\omega} \times 2^{<\omega} : f(d_1, d_2) = 1\})$ *is order-isomorphic to* $\mathbb{A}$.

We close this section with the following question. Let $\mathcal{T}$ be the first-order theory of $(\mathcal{P}(\mathbb{Q})/\sim, \leq_h)$ in the language of partially-ordered sets (i.e. the language with one binary relation symbol "$\leq$"). Is $\mathcal{T}$ decidable? Recall that not every computable partially-ordered set has decidable theory.

3 Spaces With Rank At Most 4

The set T_5 can be computed from Lemma 1. We avoid using the cumbersome, literal definition of *type* and simply list the elements of T_5 in the form i_n where $i_n \in T_{i+1} \setminus T_i$ and we list the types starting from $n = 0$ in arbitrary order (e.g. $1_0 := (\emptyset, \emptyset)$ is the unique element of $T_2 \setminus T_1 = T_2$ while $2_0 := (\{(\emptyset, \emptyset)\}, \emptyset)$ and $2_1 := (\{(\emptyset, \emptyset)\}, \{(\emptyset, \emptyset)\})$ are the elements of $T_3 \setminus T_2$). Carrying on in this manner we can compute the elements of $T_4 \setminus T_3$ by hand. They are:

$3_0 := (\{1_0, 2_0\}, \emptyset)$ $\quad$ $3_1 := (\{1_0, 2_0\}, \{1_0\})$

$3_2 := (\{1_0, 2_0\}, \{1_0, 2_0\})$ $\quad$ $3_3 := (\{1_0, 2_1\}, \{1_0\})$

$3_4 := (\{1_0, 2_1\}, \{1_0, 2_1\})$ $\quad$ $3_5 := (\{1_0, 2_0, 2_1\}, \{1_0\})$

$3_6 := (\{1_0, 2_0, 2_1\}, \{1_0, 2_0\})$ $\quad$ $3_7 := (\{1_0, 2_0, 2_1\}, \{1_0, 2_1\})$

$3_8 := (\{1_0, 2_0, 2_1\}, \{1_0, 2_0, 2_1\})$

So we have $|T_4| = 12 = 9 + 2 + 1$. A simple computer program (about 75 lines of code) can be run to list all the types in T_5. We find that $|T_5| = 20106 = 20094 + 9 + 2 + 1$, though beyond this there is no hope of listing all the types since there are probably about 10^{5000} types in T_6. Let us take stock of the situation. We know that all sufficiently small neighborhoods of a fixed point (in one of the spaces we are studying) are homeomorphic. Furthermore, restricting our attention to spaces with rank at most 4, we see that exactly 20106 homeomorphism types arise in this manner. Next one can check by hand that the types in T_4 fit into 8 embeddability equivalence classes:

$\{1_0\}$ $\{2_0\}$ $\{2_1\}$ $\{3_0, 3_1\}$ $\{3_2\}$ $\{3_3, 3_5\}$ $\{3_6\}$ $\{3_4, 3_7, 3_8\}$

It is actually possible to calculate all the embeddability equivalence classes of T_5 and the ordering of those classes by hand. There are the 8 classes above, plus 31 additional classes. The author's hand calculation was verified by a computer program to check that each of the 20094 classes in $T_5 \setminus T_4$ is embeddability-equivalent to exactly one of these 31 classes. We list these classes below, numbered arbitrarily, together with the number of types in each equivalence class, and typical representatives of each class. Set brackets and commas between elements are dropped so that, for example, the type $(\{1_0, 2_0, 3_2\}, \{1_0, 2_0\})$ appears as $(1_0 2_0 3_2, 1_0 2_0)$ and "T_4" abbreviates the list of all types in T_4.

No.	Types	Representatives
0	13	$(1_02_03_0, \emptyset) \sim (1_02_02_13_03_1, 1_02_0)$
1	8	$(1_02_03_2, 1_02_0) \sim (1_02_02_13_03_13_2, 1_02_0)$
2	25	$(1_02_03_3, 1_02_0) \sim (1_02_02_13_03_13_3, 1_02_0)$
3	12	$(1_02_03_23_3, 1_02_0) \sim (1_02_02_13_03_13_23_33_5, 1_02_0)$
4	32	$(1_02_03_6, 1_02_0) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_0)$
5	6	$(1_02_02_13_0, 1_02_02_1) \sim (1_02_02_13_03_1, 1_02_02_1)$
6	4	$(1_02_02_13_2, 1_02_02_1) \sim (1_02_02_13_03_13_2, 1_02_02_1)$
7	25	$(1_02_02_13_3, 1_02_02_1) \sim (1_02_02_13_03_13_33_5, 1_02_02_1)$
8	12	$(1_02_02_13_23_3, 1_02_02_1) \sim (1_02_02_13_03_13_23_33_5, 1_02_02_1)$
9	32	$(1_02_02_13_6, 1_02_02_1) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_1)$
10	498	$(1_02_02_13_4, 1_02_02_1) \sim (T_4, 1_02_02_1)$
11	15	$(1_02_03_0, 1_02_03_0) \sim (1_02_02_13_03_1, 1_02_02_13_03_1)$
12	15	$(1_02_03_03_2, 1_02_03_0) \sim (1_02_02_13_03_13_2, 1_02_02_13_03_1)$
13	30	$(1_02_02_13_03_3, 1_02_03_0) \sim (1_02_02_13_03_13_33_5, 1_02_02_13_03_1)$
14	30	$(1_02_02_13_03_23_3, 1_02_03_0) \sim (1_02_02_13_03_13_23_33_5, 1_02_02_13_03_1)$
15	80	$(1_02_02_13_03_6, 1_02_03_0) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_13_03_1)$
16	560	$(1_02_02_13_03_4, 1_02_03_0) \sim (T_4, 1_02_02_13_03_1)$
17	27	$(1_02_03_2, 1_02_03_2) \sim (1_02_02_13_03_13_2, 1_02_02_13_03_13_2)$
18	54	$(1_02_02_13_23_3, 1_02_03_2) \sim (1_02_02_13_03_13_23_33_5, 1_02_02_13_03_13_2)$
19	72	$(1_02_02_13_23_6, 1_02_03_2) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_13_03_13_2)$
20	504	$(1_02_02_13_23_4, 1_02_03_2) \sim (T_4, 1_02_02_13_03_13_2)$
21	54	$(1_02_02_13_3, 1_02_02_13_3) \sim (1_02_02_13_03_13_33_5, 1_02_02_13_03_13_33_5)$
22	45	$(1_02_02_13_23_3, 1_02_02_13_3) \sim (1_02_02_13_03_13_23_33_5, 1_02_02_13_03_13_33_5)$
23	90	$(1_02_02_13_33_6, 1_02_02_13_3) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_13_03_13_33_5)$
24	1285	$(1_02_02_13_33_4, 1_02_02_13_3) \sim (T_4, 1_02_02_13_03_13_33_5)$
25	45	$(1_02_02_13_23_3) \sim (1_02_02_13_03_13_23_33_5, 1_02_02_13_03_13_23_33_5)$
26	45	$(1_02_02_13_23_33_6) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_13_03_13_23_33_5)$
27	630	$(1_02_02_13_23_33_4, 1_02_02_13_23_3) \sim (T_4, 1_02_02_13_03_13_23_33_5)$
28	243	$(1_02_02_13_6, 1_02_02_13_6) \sim (1_02_02_13_03_13_23_33_53_6, 1_02_02_13_03_13_23_33_53_6)$
29	1701	$(1_02_02_13_43_6, 1_02_02_13_6) \sim (T_4, 1_02_02_13_03_13_23_33_53_6)$
30	13902	$(1_02_13_4, 1_02_13_4) \sim (T_4, T_4)$

A pictorial representation of the poset $(T_5/\sim, \leq_h)$ appears in Figure 1. In this picture there is a downhill line from σ to τ only when $\sigma < \tau$ and there is no class ρ with $\sigma < \rho < \tau$. Notice that some classes in $(T_5 \setminus T_4)/\sim$ are incomparable to classes in $T_4/\sim$. There are also some fairly large antichains (e.g. $\{3, 6, 7, 11\}$).

We close by answering a question of Rehana Patel: Is $\mathbb{A}$ (equivalently $(\mathcal{P}(\mathbb{Q})/\sim, \leq_h)$) a lattice? Consider the embeddability classes of the spaces $S(3_2)$ and $S(3_3)$. These classes have no least upper bound since (the classes of) $S(3_2) \oplus$

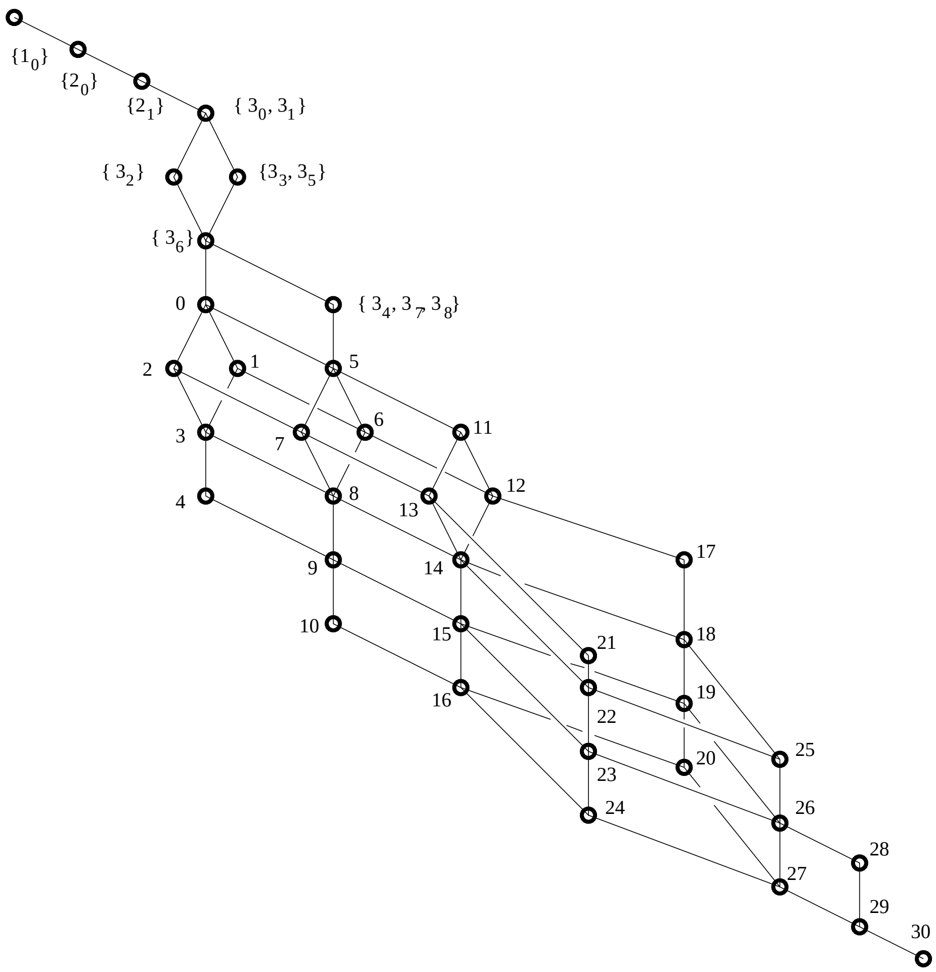

Fig. 1. Nearest Neighbor Representation of $(T_5/\sim, \leq_h)$

$S(3_3)$ and $S(3_6)$ are incomparable classes which are both minimal among all classes of spaces into which both $S(3_2)$ and $S(3_3)$ embed.

References

[1] W. D. Gillam, *Embeddability Properties of Countable Metric Spaces*, Top. App. 148 (2005) 63-82.