LAFFORGUE "BACKGROUND SEMINAR" PART 3 - UNIFORMIZATION OF Bun_G

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1. More about the moduli stack of G-bundles

Recall our setup from before: k is a field[1], X a projective connected smooth curve over k, and G an affine algebraic group over k.

We've defined a functor (really, a lax functor)

$$\begin{aligned}\mathrm{Bun}_G : (\mathbf{Sch}/k)^{\mathrm{opp}} &\to \mathbf{Gpd}\\ S &\mapsto \{G\text{-bundles on } X \times_k S, \text{ with } G\text{-bundle isomorphisms}\}.\end{aligned}$$

For us, G-bundles are taken in the fppf topology (or, equivalently, the étale topology). Remember that Bun_G depends on the choice of X, even though the notation suppresses this!

Last time, Francois proved that $\mathrm{Bun}_{\mathrm{GL}_n}$ is an algebraic stack:

- It satisfies descent in the fppf (equivalently, étale) topology; i.e., it's a stack;
- Its diagonal morphism is representable by an algebraic space (implying that *any* morphism from an algebraic space to $\mathrm{Bun}_{\mathrm{GL}_n}$ is representable);
- It possesses a smooth cover by a scheme.

Actually, we proved a bit more: $\mathrm{Bun}_{\mathrm{GL}_n}$ is locally of finite presentation over k and its diagonal is of finite presentation and is represented by an affine scheme. The moduli stack is *not*, however, quasicompact, as it has infinitely many components, indexed by degree.

We can use this result to prove the same thing for arbitrary G:

Proposition 1.1. *For any G as above, Bun_G is an algebraic stack.*

Proof. Choose an embedding $G \hookrightarrow \mathrm{GL}_n$. This induces a morphism of stacks

$$\mathrm{Bun}_G \to \mathrm{Bun}_{\mathrm{GL}_n}.$$

It suffices to show that this morphism is representable by a scheme (this is easy; for example, see Stacks Project, Tag 05UM). So take a scheme S and a map $S \to \mathrm{Bun}_{\mathrm{GL}_n}$; i.e., a GL_n-bundle F over $X \times_k S$. The pullback $\mathcal{X} := \mathrm{Bun}_G \times_{\mathrm{Bun}_{\mathrm{GL}_n}} S$ is what we want to show is a scheme.

By unwinding definitions, we find that $\mathcal{X}$ is the functor

$$\begin{aligned}\mathcal{X} : (\mathbf{Sch}/S)^{\mathrm{opp}} &\to \mathbf{Gpd}\\ S' &\mapsto \{\text{pairs } (E, \alpha)\text{: } E \text{ a } G\text{-bundle on } X \times_S S', \alpha \text{ an isomorphism}\\ &\qquad E(\mathrm{GL}_n) \xrightarrow{\sim} F \times_S S', \text{ with isomorphisms of pairs}\}.\end{aligned}$$

[1] Sorger calls for an algebraically closed field in his notes, but it is unnecessary for everything we will do as well as uncomfortably restrictive from the point of view of function field Langlands.

Here $E(\mathrm{GL}_n)$ is the "extension of structure group" which promotes the G-bundle E to a GL_n-bundle via the chosen embedding. We can check easily that any such isomorphism of pairs is necessarily unique (because $G \hookrightarrow \mathrm{GL}_n$ is an embedding), and therefore this functor actually maps to **Set**.

Now note that a pair (E, α) is the same information as a section of $F(\mathrm{GL}_n/G) \times_S S'$ over $X \times_S S'$. To show this, consider the following cartesian diagram:

$$\begin{array}{ccc} \sigma^*(F \times_S S') & \longrightarrow & F \times_S S' \\ \downarrow & & \downarrow \\ X \times_S S' & \xrightarrow{\sigma} & F(\mathrm{GL}_n/G) \times_S S' \end{array}$$

Given σ, the top arrow of the diagram is a map from a G-bundle over $X \times_S S'$ to $F \times_S S'$ which is an isomorphism upon extending the structure group to GL_n, and given a pair (E, α) we can go backwards and construct a section σ. A tedious check shows that this bijection is in fact functorial, so it suffices to prove representability of such sections.

By the theory of Hilbert schemes, representability follows if we can prove that the map $F(\mathrm{GL}_n/G) \to X$ is quasiprojective. This follows from a theorem of Chevalley[2]: there exists a finite-dimensional representation V of GL_n and a line ℓ of V such that G is the stabilizer of ℓ. This gives us an embedding $\mathrm{GL}_n/G \hookrightarrow \mathbb{P}(V^*)$, and working globally on X, an embedding $F(\mathrm{GL}_n/G) \hookrightarrow \mathbb{P}(F(V^*))$. This proves quasiprojectivity and hence the theorem. □

In fact, the above argument can easily be refined to show that Bun_G is as nice as $\mathrm{Bun}_{\mathrm{GL}_n}$: it is locally of finite presentation over k and its diagonal is of finite presentation and is represented by an affine scheme.

Partially for future use and partially just to prove that Bun_G is a reasonable object to work with, I make the following three claims (with sketchy or nonexistent proofs):

Proposition 1.2. *Bun_G is smooth.*

Proof. To show this, one first shows that a valuative criterion for smoothness holds for stacks locally of finite presentation over a field (say): that is; Bun_G is smooth if and only if for every valuation ring A and square-zero ideal I, we can fill in the following diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & \mathrm{Bun}_G \\ \downarrow & \overset{\exists ?}{\nearrow} & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

That is, we have to lift G-bundles on $\mathrm{Spec}(A/I)$ to G-bundles on $\mathrm{Spec}(A)$. After some deformation theory, it turns out that the obstruction to lifting a vector bundle lies in an Ext^2 group, which vanishes because X has dimension one. □

[2]This important theorem is used to construct schematic quotients of algebraic groups. A quick proof: consider the adjoint representation of GL_n, and the ideal of G-invariants $I(G)$ inside the group ring $\mathbb{C}[\mathrm{GL}_n]$. It is clear that G is the stabilizer of $I(G)$. By noetherianity, choose a finite-dimensional vector space W generating $I(G)$, which we may assume is G-stable. Pick a finite-dimensional GL_n-stable vector space U containing W. Now to force W to be a line, just take the nth exterior power, where $n = \dim W$; i.e., let $V = \wedge^n U$ and $\ell = \wedge^n W$.

This may be somewhat unsatisfying: the smoothness of Bun_G came down to the fact that we are working with G-bundles over a curve and not, say, a surface. Unfortunately, if $\dim X > 1$, smoothness really does fail. Igusa showed that if $\mathrm{char}(k) > 0$, there exists a smooth surface X/k such that $\mathrm{Pic}_{X/k}$ is not reduced, hence certainly not smooth. Via the smooth surjective morphism $\mathrm{Bun}_{\mathbb{G}_m} \to \mathrm{Pic}_{X/k}$ that forgets automorphisms (the fibers are all copies of $\mathrm{B}\mathbb{G}_\mathrm{m}$), we see that $\mathrm{Bun}_{\mathbb{G}_m}$ cannot possibly be smooth either.

Proposition 1.3. *If G is reductive, then Bun_G has constant dimension $\dim(G)\cdot(g-1)$, where g is the genus of X.*

Proof. This follows by looking at the cotangent complex at each point of Bun_G. Because G is reductive, there is an isomorphism of Lie groups $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, which allows us to calculate the Euler characteristic of the cotangent complex via Riemann-Roch. This Euler characteristic is exactly minus the dimension of Bun_G at the point in question.[3] □

Note that if $g = 0$ and $\dim(G) > 0$, Bun_G has negative dimension.

Proposition 1.4. *We can calculate the ℓ-adic cohomology of $\mathrm{Bun}_{\mathrm{GL}_n}$ precisely as a ring.*

Proof. For a sketch of the proof, see section 4.7 of Heinloth's notes; the ring in question is freely generated by certain explicit classes (the "Atiyah-Bott classes"). For general reductive G with small restrictions on characteristic, essentially the same formula is true, due to Heinloth and Schmitt. □

2. Uniformization of the moduli stack of G-bundles

In the process of stating the uniformization result for Bun_G, we'll see why we care about Bun_G (and sheaves on it) in the Langlands program, and the result itself will allow us to start constructing explicit sheaves on Bun_G. In particular, in many cases we can calculate $\mathrm{Pic}_{\mathrm{Bun}_G}$ explicitly. Along the way, we'll meet the *affine Grassmannian*, which is necessary to set up the geometric Satake isomorphism of Mirković and Vilonen.

2.1. **Warm-up example 1.** This example is taken from Frenkel, 3.2, and is originally due to A. Weil. Assume that $k = \mathbb{C}$ for simplicity (this is not necessary for the result, but the existence of the analytic topology will make our lives slightly easier). Let F be the function field of X. We can define the adèles $\mathbb{A}_F$ and integer adèles $\mathcal{O}_F$ just as if F were a global field. Then the claim is that there is a bijection of sets as follows:

$$\{\text{isomorphism classes of } \mathrm{GL}_n\text{-bundles on } X\} \leftrightarrow \{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(\mathcal{O}_F)\}.$$

Of course, GL_n-bundles are the same as holomorphic vector bundles in this setting.

Here's the outline of a proof. First, I claim that any vector bundle V on X can be trivialized on the complement of finitely many points. This follows from the existence of $\dim V$ meromorphic sections of V which are linearly independent at all but finitely many points, which follows from an easy induction on dimension.

[3] In fact, one can take this as the definition of the dimension of an algebraic stack, rather than by using a presentation. For details, see Sam Raskin's notes.

So say that V can be trivialized on $X \setminus \{x_1, \dots, x_N\}$. It can also clearly be trivialized on small discs (for the analytic topology) D_{x_i} around each x_i. Thus V is determined by the transition functions between these two trivializations on their intersections, which are the *punctured discs* $D_{x_i}^\times$, $1 \le i \le N$. If we use $\mathrm{GL}_n(U)$ to denote GL_n-valued holomorphic functions on $U \subset X$, then these transition functions are elements of

$$T = \prod_{i=1}^N \mathrm{GL}_n(D_{x_i}^\times).$$

However, two sets of transitions functions give rise to the same V if we pick a new trivialization on $X \setminus \{x_1, \dots, x_N\}$, which amounts to multiplication on the left of T by $\mathrm{GL}_n(X \setminus \{x_1, \dots, x_N\})$; likewise, we get the same V if we pick a new trivialization on the D_{x_i}, which amounts to a multiplication on the right of T by $\prod_{i=1}^N \mathrm{GL}_n(D_{x_i})$. After quotienting out these groups we really have nailed down V, so we get a bijection of sets

$$\begin{aligned}&\{\text{isomorphism classes of } \mathrm{GL}_n\text{-bundles on } X \text{ trivial on } X \setminus \{x_1, \dots, x_N\}\} \\ &\leftrightarrow \left\{ \mathrm{GL}_n(X \setminus \{x_1, \dots, x_N\}) \backslash \prod_{i=1}^N \mathrm{GL}_n(D_{x_i}^\times) / \prod_{i=1}^N \mathrm{GL}_n(D_{x_i}) \right\}.\end{aligned}$$

Now, using the strong approximation theorem, we pass to "formal discs;" i.e., instead of $\mathrm{GL}_n(D_{x_i})$ we consider $\mathrm{GL}_n(\mathbb{C}[[t_{x_i}]])$, where t_{x_i} is a local parameter at x_i. Similarly, the punctured formal disc is constructed by removing the maximal ideal, which amounts to taking the fraction field of the complete local ring, so $\mathrm{GL}_n(D_{x_i}^\times)$ corresponds to $\mathrm{GL}_n(\mathbb{C}((t_{x_i})))$. Finally, we consider the union over all possible finite sets $\{x_1, \dots, x_N\}$, getting

$$\begin{aligned}&\{\text{isomorphism classes of } \mathrm{GL}_n\text{-bundles on } X\} \\ &\leftrightarrow \left\{ \mathrm{GL}_n(F) \backslash \prod_{x \in X}{}' \mathrm{GL}_n(\mathbb{C}((t_{x_i}))) / \prod_{x \in X} \mathrm{GL}_n(\mathbb{C}[[t_{x_i}]]) \right\} \\ &= \{\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F) / \mathrm{GL}_n(\mathcal{O}_F)\}.\end{aligned}$$

This gives us a hint of the sought-after connection with the Langlands program: recall that *unramified* automorphic representations on GL_n possess a unique spherical function on $\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F) / \mathrm{GL}_n(\mathcal{O}_F)$, and conversely such functions give rise to automorphic representations[4]. Therefore the data of automorphic representations might have a reformulation in terms of data on the space of bundles over X; specifically, according to Grothendieck's function-sheaf dictionary, they should have incarnations as certain (perverse) *sheaves* on $\mathrm{Bun}_{\mathrm{GL}_n}$. In order to make this more precise, we will require something better than a bijection of sets (and it would be best to allow a wider range of algebraic groups).

2.2. **Warm-up example 2.** This one is easier than the last, but is notationally suggestive. Now we work in the topological setting. For this subsection only, let X be a smooth compact connected oriented real surface, let G be any connected

[4] something similar holds for ramified automorphic representations of GL_n: each such possesses a spherical function on $\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F) / \mathrm{GL}_n(K)$ for some compact subgroup K

topological group, and let "G-bundles" refer to topological G-bundles. Pick a point $x \in X$, a neighborhood D of x_0, and define the following "loop groups:"

$$L^{\mathrm{top}}G = \{\text{continuous maps } D \setminus \{x\} \to G\},$$
$$L_+^{\mathrm{top}}G = \{\text{continuous maps } D \to G\},$$
$$L_X^{\mathrm{top}}G = \{\text{continuous maps } X \setminus \{x\} \to G\}.$$

All three inherit group structures from the group law on G, and we have homomorphisms $\iota_1 : L_X^{\mathrm{top}}G \to L^{\mathrm{top}}G$ and $\iota_2 : L_+^{\mathrm{top}}G \to L^{\mathrm{top}}G$ by restriction. Let $\mathrm{Bun}_G^{\mathrm{top}}$ denote the set of isomorphism classes of G-bundles on X.

Then the claim is that we have a bijection of sets as follows:

$$Bun_G^{\mathrm{top}} \leftrightarrow \iota_1(L_X^{\mathrm{top}}G)\backslash L^{\mathrm{top}}G/\iota_2(L_+^{\mathrm{top}}G).$$

The proof is almost immediate, following the ideas of the first example; the hypotheses on G and X ensure that all G-bundles on X can be trivialized on $X \setminus \{x\}$ already, and gluing of topological bundles is obvious.

2.3. **The uniformization theorem.** Now here's the real deal. Choose a closed point $x \in X$. As discussed above, the algebraic analogue to a disc around x is a formal neighborhood, $D_x = \mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$, and the punctured disc is constructed by localizing away the offending maximal ideal to get $D_x^\times = \mathrm{Spec}(\mathrm{Frac}(\hat{\mathcal{O}}_{X,x}))$. In general, if $U = \mathrm{Spec}R$ is an affine scheme, set $D_U = \mathrm{Spec}(R[[z]])$ and $D_U^\times = \mathrm{Spec}(R((z)))$, and let $X^\times = X \setminus \{x\}$ with the obvious scheme structure and let $X_U^\times = X^\times \times_k U$ be the base change.

Define "loop group" functors as follows, where $U = \mathrm{Spec}R$:

$$LG, L_+G, L_XG : (\mathbf{Aff}/k)^{\mathrm{opp}} \to \mathbf{Grp}$$
$$LG : U \mapsto \mathrm{Hom}(D_U^\times, G) = G(R((z))),$$
$$L_+G : U \mapsto \mathrm{Hom}(D_U, G) = G(R[[z]]),$$
$$L_XG : U \mapsto \mathrm{Hom}(X_U^*, G) = G(\mathcal{O}(X_U^*)).$$

As these functors are easily checked to be sheaves for the fppf topology on **Sch**, defining them on affine schemes uniquely specifies them on all schemes. We can view L_+G as a subfunctor of LG, and therefore let

$$\mathcal{G}r_G = LG/L_+G$$

be the fppf-sheafification of the quotient functor. This functor is the *affine Grassmannian*, and it actually turns out to be an ind-scheme. The functor L_XG acts on $\mathcal{G}r_G$ on the left, so we can form the quotient fppf-stack $[L_XG\backslash\mathcal{G}r_G]$. The uniformization theorem is the claim that, for semisimple G, there exists a canonical isomorphism of stacks

$$\mathrm{Bun}_G \xrightarrow{\sim} [L_XG\backslash\mathcal{G}r_G].$$

In particular, the affine Grassmannian $\mathcal{G}r_G$ is an L_XG-bundle over Bun_G that is locally trivial in the fppf topology. These statements are not true for groups G that are not semisimple; for example, line bundles on a curve cannot be trivialized on the complement of a point in any reasonable topology that comes from algebraic geometry.

The proof, which does require some actual work, follows the same outline as the proofs above: we need to trivialize G-bundles over $X_U^\times$ locally in the fppf topology (that is, after a base changes of the field k by an fppf cover), and we need to glue them in the fppf topology to trivial bundles over D_U to get bundles on X_U.

The first step is a theorem of Drinfeld and Simpson; in fact, we get the further fact that if $\mathrm{char}(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$ then we can trivialize in the étale topology. There is a generalization of this result to certain other group schemes by Heinloth. The second step is a theorem of Beauville and Laszlo; the gluing result is not trivial because for arbitrary (non-noetherian) affine U the induced map $D_U \to X_U$ may not be flat, so we cannot simply apply fppf descent. Once these two theorems have been proven, the rest is entirely formal.

In the immediate future, we will use the uniformization result to construct certain line bundles on Bun_G by constructing $L_X G$-equivariant line bundles on the affine Grassmannian.

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