p-ADIC UNIFORMIZATION OF CURVES

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1. Introduction: why bother?

Let us presume that we have at our disposal the fully-formed theory of rigid-analytic spaces., as sketched in my last talk. Why would we care to look for uniformizations of algebraic objects in the rigid analytic category?[1]

Let's look at the analogous situation in the complex-analytic category. We know that, for example, abelian varieties are uniformized by spaces of the form $\mathbb{C}^g/\Lambda$, where Λ is a free $\mathbb{Z}$-module of full rank (i.e., a lattice). What does this gain us? Here are two examples: complex uniformization allows us to "see" the structure of the torsion points explicitly (in an obvious way), and also allows us to "see" endomorphisms of abelian varieties (by looking at the endomorphism ring $\mathrm{End}(\mathbb{C}^g) = M_g(\mathbb{C})$ and picking out those which respect the action of Λ).

In the p-adic setting, we can do all this and more. Namely, we have the important additional fact that p-adic uniformization is Galois-equivariant (this has no content over $\mathbb{C}$, of course), so we can use uniformization to study, for example, $\mathrm{Gal}_{\mathbb{Q}_p}$ and its action on the Tate module. It turns out that p-adic uniformization can also be used to construct explicit Néron models and has significant computational advantages (see Kadziela's article, for example).

2. The Tate curve

Consider the standard complex uniformization of elliptic curves: given any $E/\mathbb{C}$, there exists a lattice Λ in $\mathbb{C}$ such that

$$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$$

as complex-analytic spaces. If we try the same thing over, say, $\mathbb{C}_p = \hat{\overline{Q_p}}$, we run into difficulties immediately. In fact, for *any* free $\mathbb{Z}$-module of finite rank in $\mathbb{C}_p$, it is true that $\mathbb{C}_p/\Lambda \simeq \mathbb{C}_p$ as rigid spaces. Similar ideas are quite fruitful lead to the theory of Drinfeld modules, but this is not what we want (of course, $\mathbb{C}_p$ cannot be the analytification of an elliptic curve for any number of reasons: look at the torsion points, or note that it is not a proper rigid space while the analytification of an elliptic curve is).

So we make a modification: going back to the complex case, make the change of variables $e : \mathbb{C} \to \mathbb{C}^*$ sending $z \mapsto e^{2\pi i z}$. This gives an isomorphism of complex spaces $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$. In getting $\mathbb{C}^*$, we've already "modded out by 1," so to get an elliptic curve we only need to quotient by some $q = e(\tau)$, where τ is $\mathbb{R}$-linearly independent of 1. This alternate complex uniformization is

$$E(\mathbb{C}) \simeq \mathbb{C}^*/\langle q \rangle$$

[1] Of course, historically, uniformization of the Tate curve was what *led* to the development of rigid analysis, but for the sake of exposition...

for some $q \in \mathbb{C}^*$. It is this that admits a p-adic analogue. Namely, we want to show that for any complex non-archimedean field k (which we fix for the remainder of the talk) and any $q \in k^*$ such that $0 < |q| < 1$, the *Tate curve*

$$T = \mathbb{G}_{m,k}^{\mathrm{an}}/\langle q \rangle$$

is the analytification of a unique elliptic curve. We also want to describe exactly which elliptic curves arise in this way.

First, we really should define T more precisely, which we can do using the following theorem:

Theorem 2.1 (Existence of quotients). *Let Y be a rigid analytic space and Γ a group of automorphisms of Y that acts discontinuously (meaning: there exists an admissible affinoid cover $\{Y_i\}$ of Y such that $\{\gamma \in \Gamma : \gamma Y_i \cap Y_i \neq \emptyset\}$ is finite for each i). Then there exists a rigid analytic space Y/Γ and a morphism $p : Y \to Y/\Gamma$ such that for any admissible open Γ-invariant $U \subset Y$ and any Γ-invariant morphism $f : U \to X$, the set $p(U)$ is an admissible open and f factors uniquely through $p(U)$ (that is, there exists a unique morphism $g : p(U) \to X$ such that $f = g \circ p$).*

Sketch. As a set, Y/Γ is the set-theoretic quotient and p the quotient map. As a G-topological space, a set $U \subset Y/\Gamma$ is admissible if $p^{-1}(U)$ is, and a cover $\{U_i\}$ of U is an admissible cover if $\{p^{-1}(U_i)\}$ is an admissible cover of $p^{-1}(U)$. Finally, the structure sheaf is defined as $\mathcal{O}_{Y/\Gamma}(U) = (\mathcal{O}_Y(p^{-1}(U)))^{\Gamma}$. It is a straightforward exercise to check that this construction actually gives a rigid analytic space that satisfies the conditions of the theorem. □

Next, we will need to know that T has some nice properties.

Proposition 2.2. *The Tate curve T is a connected, nonsingular, separated, and proper rigid space of dimension 1.*

Before the proof, we need to define a couple of these words. A rigid analytic space is *separated* if the diagonal morphism is a closed immersion (i.e., defined by a coherent sheaf of ideals). It turns out (and this is the criterion we will use) that a space X over k is separated if and only if there is an admissible affinoid cover $\{X_i\}$ of X such that all intersections $X_i \cap X_j$ are affinoid and the canonical maps $\mathcal{O}_X(X_i)\hat{\otimes}_k\mathcal{O}_X(X_j) \to \mathcal{O}_X(X_i \cap X_j)$ are surjective.

A rigid space X is *proper* if there exist two finite admissible affinoid covers $\{X_i\}_{1\leq i\leq N}$ and $\{X_i'\}_{1\leq i\leq N}$ such that X_i lies in the interior of X_i' for all i. By this last bit, we mean more precisely that if $X_i' = \mathrm{Sp}(A)$, $A \simeq k\langle z_1, \ldots, z_n\rangle/(f_1, \ldots, f_m)$, and $\overline{z_i}$ denotes the image of z_i in A, then there exists a $\rho < 1$ such that $X_i \subset \{x \in X_i' : \text{all } |\overline{z_i}(x)| \leq \rho\}$.

Proof of proposition. It is easy to check that T has dimension 1, is nonsingular, and is connected.

In order to check properness and separatedness, let's write down an explicit cover of T. Recalling that $\mathbb{G}_{m,k}^{\mathrm{an}} = \mathrm{MaxSpec}(k[z, z^{-1}])$, we let

$$U_0 = p\left(\{x \in \mathbb{G}_{m,k}^{\mathrm{an}} : |q|^{1/3} \leq |z(x)| \leq |q|^{-1/3}\}\right),$$
$$U_1 = p\left(\{x \in \mathbb{G}_{m,k}^{\mathrm{an}} : |q|^{2/3} \leq |z(x)| \leq |q|^{1/3}\}\right).$$

It is easy to check that these are two admissible open sets that give an admissible open cover of T. They intersect in two disjoint sets, the "ring at $|q|^{2/3}$," which we will call U_-, and the "ring at $|q|^{1/3}$," which we will call U_+. Then $U_0 \cap U_1 = U_- \coprod U_+$.

Properness is easy: choose slightly larger ring domains U_0' and U_1' such that U_0 is in the interior of U_0' and U_1' is in the interior of U_1' (hence the intersection of U_0' and U_1' will be the disjoint union of two annuli). It is similarly easy to check that $\{U_0', U_1'\}$ is an admissible cover of T. Properness follows by definition.

To show that T is separated, we need to show that

$$\mathcal{O}_T(U_0)\hat{\otimes}_k\mathcal{O}_T(U_1) \to \mathcal{O}_T(U_0 \cap U_1)$$

is surjective. We can do this by explicitly identifying these rings as sets of convergent power series and calculating, but I will omit the precise details. For example, we have

$$\mathcal{O}_T(U_0) \simeq \left\{ \sum_{n\geq 0} a_n(\pi z)^n + \sum_{n>0} b_n \left(\frac{\pi}{z}\right)^n : \lim a_n = \lim b_n = 0 \right\},$$

where $\pi^3 = q$ (possibly going to a finite extension of k, which we can do). Similarly we can explicitly write down the other two rings and the map, checking that it is surjective. This is one of two places that we actually have to make a calculation about T. □

If we knew that T were projective, we'd be essentially done, as we could then use GAGA (and complete an easy calculation). But we don't, and there are several ways that we could proceed. We could proceed (like in Fresnel and van der Put) largely as in the complex case, by bootstrapping our way using divisors to construct an ample line bundle yielding a map into projective space, which we then have to argue is an embedding. This last step proceeds by first showing that the map is injective and injective on tangent spaces, even when going up to the completion of the algebraic closure. That this suffices requires the use of the Gerritzen-Grauert theorem and GAGA, so it is harder than in the complex case. The advantage is that the embedding criterion is more general; it makes no reference to the fact that we are starting with a curve.

Alternatively, we could try to appeal more directly to a rigid-analytic Chow's lemma, which would imply that any proper nonsingular rigid curve is automatically projective. We will take this second route and prove this consequence of Chow's lemma as a corollary of the following (for a reference, see Mitsui's paper):

Theorem 2.3 (Rigid-analytic Riemann-Roch)**.** *Let X be a proper separated nonsingular rigid space of dimension 1. Then*

$$\chi(\mathcal{L}(D)) = \chi(\mathcal{O}_X) + \deg D$$

for any divisor D on X.

Of course, we have to define what we mean by all this; fortunately, everything proceeds in the same way as in the algebraic (or complex) case. A *divisor D* on X is a finite formal sum

$$D = \sum_{i=1}^{s} n_i[x_i], \quad n_i \in \mathbb{Z}, x_i \in X,$$

and we set

$$\deg D = \sum n_i \dim_k(\mathcal{O}_{x_i}/\mathfrak{m}_{x_i}).$$

We form the sheaf of meromorphic functions $\mathcal{M}$ on X in the obvious way (that is, for a regular affinoid subset $R = \mathrm{Sp}(A)$, set $\mathcal{M}(R)$ as the total ring of fractions of A; these glue precisely because X is separated).For any $f \neq 0 \in \mathcal{M}(X)$, let

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f)[x].$$

To any divisor D we associated the sheaf $\mathcal{L}(D)$ of $\mathcal{O}_X$-modules defined by

$$\mathcal{L}(D)(U) = \{f \in \mathcal{M}(U) : \mathrm{div}(f) \geq -D|_U\}.$$

Finally, as usual, for any coherent sheaf $\mathcal{F}$ on X we associate the Euler characteristic

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

A basic (but hard, and employing some nontrivial functional analysis!) result of Kiehl in coherent cohomology of rigid spaces implies that these dimensions are finite and equal to zero for i large enough; hence $\chi(\mathcal{F})$ is well-defined.

I will skip the proof of the Riemann-Roch theorem, but it is not any more difficult than the complex case once one knows the finiteness of the cohomology groups. Of course, one cannot simply use the algebraic Riemann-Roch theorem and GAGA; the whole point is that we do not yet know that T is projective.

Corollary 2.4. *Any nonsingular proper rigid space X of dimension 1 is projective.*

Sketch. Without loss of generality we can assume X is irreducible. We use Riemann-Roch to find a point $P \in X$ and some $n \in \mathbb{Z}$ such that $\mathcal{L}(n \cdot P)$ admits a section that is not a section of $\mathcal{L}((n-1) \cdot P)$. This yields a proper map $\phi : X \to \mathbb{P}_k^{1,\mathrm{an}}$. By Kiehl's direct image theorem for rigid spaces, $\phi(X)$ is a closed subset of $\mathbb{P}_k^{1,\mathrm{an}}$, and by the Stein factorization (which also holds in this setting, see BGR, 9.6.3, Lemma 4 and Proposition 5) we get maps $X \to S \to \phi(X)$ where S is finite over $\phi(X)$. By comparing structure sheaves it is easy to see that the first map is an isomorphism. By GAGA, the categories of finite spaces over any projective variety Y and its analytification Y^{an} are equivalent. Therefore X, being a finite space over a projective variety $\phi(X)$, must be projective. □

Now we are ready for:

Theorem 2.5 (p-adic uniformization of elliptic curves)**.** *$T \simeq E^{an}$ for some elliptic curve E/k.*

Proof. As the genus is defined by coherent cohomology, it suffices by GAGA and the above corollary to show that $g := \dim_k H^1(T, \mathcal{O}_T) = 1$. This requires another calculation, this time of the Čech cohomology of the complex

$$\mathcal{O}_T(U_0) \oplus \mathcal{O}_T(U_1) \xrightarrow{d} \mathcal{O}_T(U_0 \cap U_1).$$

Like the previous calculation, it is totally explicit but I have no time to do it at the board. □

Now, if we like, we can write down explicit formulas for an equation defining E in terms of q; this is what Tate did. I'll skip this, because it is not particularly enlightening (but it is nice to know that it can be done). We still care about *which* elliptic curves can be uniformized, however. The following theorem fills out our knowledge about the Tate uniformization.

Theorem 2.6. *Let $j(E)$ denote the usual j-invariant of an elliptic curve.*

(1) *If $T \simeq E^{an}$, then $|j(E)| > 1$.*
(2) *Conversely, if E/k is such that $|j(E)| > 1$, then there exists a Galois extension ℓ of k of degree ≤ 2 such that E_ℓ^{an} is isomorphic to a Tate curve.*
(3) *Two Tate curves T_1 and T_2 are isomorphic if and only if $q_1 = q_2$.*

Proof.

(1) We first claim that if $j(q) = \frac{1}{q} + 744 + \dots$ is the classical formula for the j-invariant of an elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ where $q = e^{2\pi i \tau}$, then the j-invariant of the Tate curve $\mathbb{G}_{m,k}^{\mathrm{an}}/\langle q \rangle$ is given by the same formula, interpreted as a converging Laurent series with coefficients in k. The proof of this consists of looking at Tate's explicit formulas for the equation defining the elliptic curve in terms of q; I do not know how to show this otherwise. Assuming this, we have $|j(q)| = |q|^{-1} > 1$, as $|q| < 1$.
(2) The equation $j(q) = j(E)$ has a unique solution if $|j(E)| > 1$, so the Tate curve $\mathbb{G}_{m,k}^{\mathrm{an}}/\langle q \rangle$ is the analytification of some elliptic curve E_q such that $j(E_q) = j(E)$. Therefore E_q is a twist of E (i.e., they are isomorphic over an algebraic closure). Twists of elliptic curves are classified by the cohomology group

$$H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Aut}(E_q \otimes_k k^{\mathrm{sep}})).$$

Because any endomorphism of a Tate curve must lift to an automorphism of $\mathbb{G}_{m,k}^{\mathrm{an}}$, it is easy to show that $\mathrm{End}(E_1 \otimes_k k^{\mathrm{sep}}) = \mathbb{Z}$, so $\mathrm{Aut}(E_q \otimes_k k^{\mathrm{sep}}) = \{\pm 1\}$. So if we take any character

$$h \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k^{\mathrm{sep}}/k), \{\pm 1\}),$$

then the fixed field of the kernel of h is the field ℓ that we want.
(3) Easy exercise. □

3. Mumford curves

Generalizing the Tate curve, we will consider a more general class of discontinuous actions on subsets of $\mathbb{P}_k^{1,\mathrm{an}}$ and see if the resulting quotient spaces uniformize a curve (and if so, which curves we can uniformize in this way!). To state the main theorem, we will need a few auxiliary constructions.

Let $\Gamma \subset \mathrm{PGL}_2(k)$ be a subgroup of the automorphisms of $\mathbb{P}_k^{1,\mathrm{an}}$. Its set of *limit points* is defined as the set of $a \in \mathbb{P}_k^{1,\mathrm{an}}$ such that there exists a $b \in \mathbb{P}_k^{1,\mathrm{an}}$ and a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\lim \gamma_n b = a$. We say that Γ is *discontinuous* if the limit points are not all of $\mathbb{P}_k^{1,\mathrm{an}}$ and the topological closure of any Γ-orbit is compact. A *Schottky group* is a finitely generated discontinuous nontrivial group which has no nontrivial elements of finite order (it is a nontrivial theorem that this implies that the group is actually free). Given a Schottky group Γ, we denote the set of limit points by L.

Example: we note that any Schottky group has at least two limit points; namely, the fixed points of any nontrivial element (as any element of $\mathrm{PGL}_2(k)$ has at least two fixed points). If these two points exhaust L, then without loss of generality we can choose $L = \{0, \infty\}$. The elements of $\mathrm{PGL}_2(k)$ with these fixed points are precisely the dilations. Γ is assumed to be finitely generated, so we have $\Gamma = \langle \gamma \rangle$, where $\gamma : z \mapsto qz$ for some $q \in k^*$, $|q| < 1$ (again without loss of generality). Then we're in the situation of the Tate curve considered before, and we know that

$$(\mathbb{P}_k^{1,\mathrm{an}} - L)/\Gamma \simeq E^{\mathrm{an}}$$

for some elliptic curve E.

From now on, assume that $|L| > 2$. Then (exercise!) L must be an infinite compact set. The following is the simplest version of the main theorem about Mumford curves:

Theorem 3.1. *Let Γ be a Schottky group, L the set of its limit points, and assume $|L| > 2$. Let $\Omega = \mathbb{P}_k^{1,an} - L$. Then:*

1. Γ *is a free group on $g > 1$ generators.*
2. Ω/Γ *is a rigid space, isomorphic to C^{an}, where C is a curve over k of genus g that has a reduction with the following properties: the reduction is reduced, each singularity is an ordinary double point étale-locally, the normalization of each irreducible component is a projective line, and on each irreducible component there lie at least three double points.*
3. *Conversely, if C' is a smooth projective curve with a reduction that is reduced, has only ordinary double point singularities étale-locally, and is such that the normalization of each irreducible component has genus zero, then there exists a Schottky group Γ' and a finite separable extension ℓ of k such that $(C \otimes_k \ell)^{an} \simeq \Omega'/\Gamma'$. Such curves C' are called* Mumford *curves.*

Another way of stating the above theorem without having to talk about extension of the ground field is the following: say that a curve has *split degenerate stable reduction* if it has a reduction whose only singular points are ordinary double points and $\overline{k}$-rational nodes with $\overline{k}$-rational branches, any genus zero component has ≥ 3 double points, and the normalizations of all components have genus zero. Then Ω/Γ has split degenerate stable reduction and conversely all curves with split degenerate stable reduction are expressible in that form.

The proof of the above theorem actually gives us more, in that we can explicitly construct the reduction on the analytic side of things and compute its intersection graph directly from Γ.

I lack the time to say anything meaningful about the proof of the above theorem, but here are a few ideas that go into it. We first define a canonical analytic reduction map $\Omega \to Z$, where Z is locally finite (but globally quite large, with infintiely many components). The intersection graph of Z is a tree M (related to the Bruhat-Tits tree of PGL_2) acted on by Γ. Then we study the reduction

$$r : \Omega/\Gamma \to Z/\Gamma,$$

where the intersection graph of Z/Γ is the finite graph M/Γ. By studying M/Γ and using techniques similar to those we used above to "recognize" analytifications of projective varieties, we show that r is the analytification of an algebraic reduction $C \to \overline{C}$, where $\overline{C}$ is a split degenerate stable reduction.

4. Addendum: a little about reduction of rigid spaces

Since the Mumford curves are classified by their reduction behavior (which is more complicated than just the j-invariant if $g > 1$!), I really should say something about rigid-analytic reductions. First let's consider an affinoid space $X = \mathrm{Sp}(A)$, for which there is a canonical reduction. Namely, define

$$A^\circ = \{a \in A : ||a||_{\mathrm{sp}} \le 1\},$$
$$A^{\circ\circ} = \{a \in A : ||a||_{\mathrm{sp}} < 1\},$$
$$\overline{A} = A^\circ / A^{\circ\circ},$$

where $||\cdot||_{\mathrm{sp}}$ is the spectral norm. Then $\overline{A}$ is a reduced $\overline{k}$-algebra of finite type, and we let $\overline{X}^c = \mathrm{MaxSpec}(\overline{A})$.[2] We have a *reduction map* $\mathrm{Red} : X \to \overline{X}^c$ defined as follows: for any $x \in X$ we have a surjective map $\phi : A \to A/\mathfrak{m}_x = \ell$, where ℓ is a finite extension of k. The valuation on k uniquely extends to ℓ, so we get an induced map $\overline{\phi} : \overline{A} \to \overline{\ell}$. We then set $\mathrm{Red}(x) = \ker \overline{\phi}$. It is easy to show that if U is an open affine subset of $\overline{X}^c$, then $\mathrm{Red}^{-1}(U)$ is an affinoid subset of X.

As an example, if $A = k\langle T\rangle$, then $\overline{A} \simeq \overline{k}[T]$. For simplicity, let k be algebraically closed. Then a point of $\mathrm{Sp}(A)$ "is" just an element $t \in k$ such that $|t| \le 1$, and its reduction "is" the point $\overline{t} \in \overline{k}$. In particular, if $|t| < 1$ then $\mathrm{Red}(t) = 0$.

If X is not affinoid, we need to specify more data in order to nail down a specific reduction map. For simplicity, we assume that X is reduced. A *pure affinoid covering* $U = \{X_i\}$ of X is an admissible affinoid covering that satisfies the following conditions:

(1) The spectral norm on $\mathcal{O}(X_i)$ takes its values in $|k|$.
(2) For each i, there are only finitely many j such that $X_i \cap X_j \ne \emptyset$.
(3) If $X_i \cap X_j \ne \emptyset$, then there exists an open affine $U_{i,j} \subset \overline{X_i}^c$ such that $\mathrm{Red}_{X_i}^{-1}(U_{i,j}) = X_i \cap X_j$.
(4) The natural map $\mathcal{O}_X(X_i)^\circ \otimes_{k^\circ} \mathcal{O}_X(X_j)^\circ \to \mathcal{O}_X(X_i \cap X_j)^\circ$ is surjective.

Given this data (X, U), we can form a reduced, separated variety $\overline{(X,U)}$ over $\overline{k}$ by gluing the $\overline{X_i}^c$, and a map $\mathrm{Red} : X \to \overline{(X,U)}$ by gluing the maps $\mathrm{Red}_{X_i} : X_i \to \overline{X_i}^c$. A morphism $(X,U) \to (Y,U)$ of these objects is a rigid morphism preserving open sets of the form $\mathrm{Red}^{-1}(U)$, where U is a Zariski-open set in $\overline{(X,U)}$.

As an almost-aside, this extra data is essentially equivalent to the extra data of a formal scheme over the ring of integers. More precisely, if k is discrete there is an equivalence of categories between pairs (X,U) as defined above and separated reduced flat formal schemes which are locally of finite type over k°. The reduction map of the rigid space then corresponds to the specialization map on the formal scheme.

Let;s work through a couple of examples of reductions. Take $X = \mathbb{P}_1^{k,\mathrm{an}}$ and let $U = \{X_1, X_2\}$, where $X_1 = \{z : |z| \le 1\}$ and $X_2 = \{z : |z| \ge 1\}$. It is an easy check that this is a pure affinoid covering (in particular, $\mathrm{Red}^{-1}(\mathbb{A}^1_{\overline{k}} - \{0\}) = X_i \cap X_j$, and $\mathbb{A}^1_{\overline{k}} - \{0\}$ is an affine open), and each X_i reduces to $\overline{X_i}^c = \mathbb{A}^1_{\overline{k}}$. The reductions glue together in such a way that $\overline{(\mathbb{P}^{1,\mathrm{an}}_k, U)} = \mathbb{P}^1_{\overline{k}}$.

For a less nontrivial example, consider the same rigid space with the cover $V = \{Y_1, Y_2, Y_3\}$, where $Y_1 = \{z : |z| \le |\pi|\}$, $Y_2 = \{z : |\pi| \le |z| \le 1\}$, and $Y_3 = \{z : 1 \le |z|\}$, and $\pi \in k$ is some element such that $|\pi| < 1$. Again, it is easy to check that

[2] If we were working with the more rubust *adic spaces* framework of Huber rather than rigid spaces, then we would be able to define a reduction to $\mathrm{Spec}(\overline{A})$ and thus give a more direct link to modern algebraic geometry, but we will have to content ourselves with MaxSpec here.

this is a pure affinoid cover. This time, the reduction $\overline{(\mathbb{P}_k^{1,\mathrm{an}}, V)}$ is isomorphic to two copies of $\mathbb{P}^1_{\overline{k}}$ that intersect at a single ordinary double point.

The reductions used to prove the main theorem about Mumford curves are induced by a certain infinite pure affinoid cover of Ω canonically induced by Γ. It turns out that the resulting reduction is quite large (as mentioned above, its intersection graph is an infinite tree), so the analysis involved is occasionally somewhat subtle. For details, see the book of Fresnel and van der Put.

5. References

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