MOTIVATION AND INTRODUCTION: JACQUET-LANGLANDS SEMINAR

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1. Motivation

Like all reasonable subjects in mathematics, this one tends to supply its own intrinsic motivation after a while. But since that's not very satisfying, here's one motivational path to follow.

We're presumably all number theorists, so we care about the following mysterious group:

$$G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

In general, a great way to study groups is via their representations, especially their finite-dimensional representations, so we are led immediately to the study of Galois representations.

Additionally, lots of Galois representations tend to come "from nature." As a first example, consider the *cyclotomic character*

$$\chi_\ell : G_{\mathbb{Q}} \to \mathbb{Z}_\ell^*$$

defined by $\sigma(\zeta) = \zeta^{\chi_\ell(\sigma)}$, where ζ is any ℓ^nth root of unity. This is a one-dimensional continuous Galois representation. As a second example, $G_{\mathbb{Q}}$ acts on the torsion points of an elliptic curve, hence on the ℓ-adic Tate module (which is non-canonically isomorphic to $\mathbb{Z}_\ell^2$), so after picking a basis we get a representation

$$\rho_\ell : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_\ell).$$

This representation is continuous with respect to the profinite topology on $G_{\mathbb{Q}}$ and the obvious ℓ-adic topology on $\mathrm{GL}_2(\mathbb{Q}_\ell)$. More generally, starting with any smooth projective variety $X/\mathbb{Q}$, the group $G_{\mathbb{Q}}$ acts continuously on the ith étale cohomology group

$$\mathrm{H}^i_{\text{ét}}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell),$$

giving a finite-dimensional ℓ-adic representation. By varying ℓ, we get a whole set of Galois representations that are compatible with each other in a certain sense.

As yet another class of examples, we can consider *Artin representations*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}),$$

which are required to have an open kernel (equivalently, be continuous with respect to the *discrete* topology on $\mathrm{GL}_n(\mathbb{C})$). For example, one-dimensional Artin representations arise naturally from Dirichlet characters. Picking embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ gives a bijection between isomorphism classes of Artin representations and isomorphism classes of continuous ℓ-adic representations with open kernel, so these examples are really all of the "same form."

To all of these representations, we can associate L-functions the usual way, and analytic properties of these L-functions tell us about arithmetic properties of the underlying objects (in the same way that information about the analyticity, functional equation, and special values of the Riemann ζ-function translate into arithmetic properties of $\mathbb{Z}$, such as the prime number theorem).

The *Langlands program* helps us learn about these Galois representations (and hence the arithmetic objects that give rise to them) by conjecturing a bijection (and more!) between Galois representations and certain analytic objects called *automorphic representations*. Automorphic representations are no less mysterious than Galois representations, but they're often mysterious in different ways. What is difficult to understand on the Galois side can be easy on the automorphic side, and vice-versa. As it turns out, one can also assign L-functions to automorphic representations (at least at almost every place), and the conjectured correspondence should respect this assignment (again, at least at almost every place).

These correspondences are as yet unproven even for $\mathrm{GL}_2(\mathbb{Q})$. The progress that has been made, however, underlies a lot of the great results in number theory of the last few decades, such as the modularity theorem/Fermat's Last Theorem and the resolution of the Sato-Tate conjecture.

2. A "precise" statement

At this point, it might be helpful to have a precise statement in mind, in order to understand more concretely what is going on (even if I don't define all the words). This statement is taken from Taylor's 2002 ICM article, and applies only to $\mathrm{GL}_n(\mathbb{Q})$: nothing about functoriality, or L-groups, or L-packets!

Fix an embedding $\mathbb{Q} \to \mathbb{C}$, an integer $n \geq 1$, and a multiset of integers H of cardinality n. Then there is a bijection between the following sets that preserves L-functions at almost every place:

- (AF) Irreducible subrepresentations of the right regular representation on $\mathcal{A}_H^\circ(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))$, the space of cuspidal automorphic functions with infinitesimal character indexed by H.
- (LF) Almost-everywhere equivalence classes of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}^{\mathrm{fin}}) \times (\mathfrak{gl}_n, O(n))$-modules with infinitesimal character indexed by H, central character trivial on $\mathbb{Q}^*$, and such that the L-function *and its twists by characters* satisfy certain analytic properties (e.g. analytic continuation, boundedness in vertical strips, functional equation).
- (ℓ-R) Given a fixed inclusion $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$, isomorphism classes of irreducible finite-dimensional ℓ-adic representations $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ that are de Rham at ℓ with Hodge-Tate weights H.
- (G) Irreducible ℓ-adic representations $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ that arise as subquotients of the natural representation of $G_\mathbb{Q}$ on $\mathrm{H}^i_{\text{ét}}(X \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell(j))$ for some smooth projective variety $X/\mathbb{Q}$ and integers $i > 0$, j, and for which the local representation at $p = \ell$ is de Rham with Hodge-Tate numbers H.[1] Here the presence of the integer j indicates a *Tate twist*; i.e., that one should take the tensor product with the jth power of the ℓ-adic cyclotomic character.

If one unwinds all of the definitions, one finds that the $n = 1$ case of these conjectures (with the "certain analytic properties" of (LF) modified slightly in

[1]The fact that a geometric Galois representation is de Rham at $p = \ell$ is automatic but nontrivial.

order to allow for the pole of the Riemann zeta function) essentially follows from the existence and basic properties of the Artin map in global class field theory. One can add at least a couple more sets to the above bijection: for example, one often works with a "compatible" set of ℓ-adic representations, and for appropriate definitions these compatible sets are also in bijection with the above.

Let's go through what is known about the relationships between these sets:

- (G)$\subset$(ℓ-R) Basically follows by definition.
- (AF)$\subset$(LF) This is completely known: L-functions of automorphic representations have nice analytic properties.
- (LF)$\subset$(AF) This is *not* completely known. Results of this type are known as *converse theorems*; they specify that the automorphic representations are precisely those with nice L-functions (for them and their twists). This is known for $n = 2$ by Weil '67 (reformulated in a more modern language by Jacquet and Langlands '70) and for $n = 3$ by Jacquet, Pietetski-Shapiro, and Shalika '79. Weaker results (where one twists by more representations than just characters) are known for all n by Cogdell and Pietetski-Shapiro '94, '99.
- (AF)$\subset$(G) This is known only if H consists of distinct integers, the automorphic representation π is essentially self-dual (that is, self-dual up to a twist by a character), and either $n \leq 2$ or there exists a place p such that the local representation π_p is square-integrable modulo the center. The basic strategy of all proofs in this direction so far is to realize representations in the cohomology of Shimura varieties. Results include Kottwitz '92, Clozel '91, and Harris-Taylor '01 (in the context of the proof of the local Langlands conjectures for GL_n).
- (ℓ-R)$\subset$(AF) This is the hardest inclusion, or at least the one about which the least is known (it is not even known for $n = 2$ in full generality). The case of *odd* Galois representations is Serre's conjecture and was proven in Khare and Wirtenberger '08. The proof of the modularity conjecture for elliptic curves over $\mathbb{Q}$ by Breuil, Conrad, Diamond, and Taylor in '01 can also be considered a result of this type.

I should say something about how all of this is "better than a bijection," which falls under the phrase *Langlands functoriality.* In short, there is a web of relationships between various algebraic groups over global fields, and these relationships should reflect natural relationships on the automorphic side. This web of relationships is made precise by the notion of a morphism of L-groups. In order to state any actual theorems or conjectures, we would need to consider other base fields or other reductive groups besides GL_n.

3. What is in Jacquet-Langlands?

By my count, there are three main results or themes of Jacquet and Langlands' 1970 book *Automorphic Forms on* GL(2).

- Local Langlands for GL_2 over any local field F, except if the residue characteristic is equal to 2. This is a weaker statement than the above bijections, as it only happens over one place. We get a bijection between irreducible admissible[2] representations of $\mathrm{GL}_2(F)$ and two-dimensional semisimple complex Weil-Deligne representations of the Weil group W_F attached to F. Here W_F is the subgroup of G_F that acts as an *integer* power of Frobenius on the residue field, and a Weil-Deligne representation is a representation with a little extra data: precisely, it is a pair (ρ, N) where ρ is a representation and N is a nilpotent endomorphism satisfying

$$\rho(\mathrm{Frob}) \cdot N \cdot \rho(\mathrm{Frob}^{-1}) = p^{-1} \cdot N,$$

where p is the residue characteristic of F. Furthermore, this bijection preserves L-factors, ϵ-factors (which appear in the functional equation of the global L-functions), and twisting by characters. This last requirement means essentially that the $n = 2$ version of the Local Langlands Correspondence is functorially consistent with local class field theory.

The method of proof in Jacquet-Langlands is by classifying a bunch of representations and matching them up by "brute force." Extensions of this result are known: the Local Langlands Correspondence was proven for GL_n over a field of positive characteristic by Laumon, Rapoport, and Stuhler '93, and for local fields of characteristic zero we have proofs by Harris and Taylor '01, Henniart '00, and Scholze '13.

- Some discussion of the global case for GL_2: what an automorphic representation is, what the Hecke algebras are, and so on. Keep in mind that the *global* Langlands correspondence is still unproven even for GL_2 over $\mathbb{Q}$, although we do have proofs for GL_2 (by Drinfeld '77) and for GL_n in general (Lafforgue '02).
- The *Jacquet-Langlands correspondence*: an example of functoriality (between various forms of GL_2)! This was apparently an afterthought in Jacquet-Langlands, but it has proven to be surprisingly useful. Let G be the multiplicative group of a quaternion algebra, so G is a twist of the split form GL_2. For global fields we get a bijection between automorphic representations of G of dimension greater than 1 and cuspidal automorphic representations of GL_2 that are square-integrable modulo the center at some place. There is also a corresponding local version. The method of proof is via matching terms in the Selberg trace formula: in particular, this is a global proof even for the local statement.

  The Jacquet-Langlands correspondence is useful in part because the spaces over which quaternionic automorphic representations live (as sections of a line bundle) are compact, unlike the case for GL_2 proper (think modular curves, which has to be compactified by adding cusps), so certain questions are easier to answer in that setting.

4. References

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[2]This is a sort of finite-dimensionality condition with respect to a maximal compact subgroup. Our representations are infinite-dimensional, but we don't want them to be *too* infinite-dimensional.

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