OVERVIEW OF RIGID ANALYTIC GEOMETRY

EVAN WARNER

This is very much a "technique/theory" talk, assuming no background in p-adic geometry, with no whiz-bang payoff at the end. I want to emphasize now that there *are* payoffs, I just won't prove them.

1. Introduction

The idea is simple: we want to develop a theory of analytic manifolds and spaces over fields equipped with an arbitrary complete valuation. Of course, it is a standard fact that such a field must be either $\mathbb{R}$, $\mathbb{C}$, or a field with a nonarchimedean valuation, so what we really mean is that we want to develop a theory of nonarchimedean analytic spaces. Doing this naïvely (i.e., defining manifolds in the same way as one does over $\mathbb{R}$ or $\mathbb{C}$) leads to serious difficulties, mostly due to the fact that the resulting topologies are totally disconnected, so we cannot use connectedness arguments and it is unclear how to develop useful cohomology theories. There are many ways of getting around this, and in this talk I'll discuss the (chronologically) first method, due to Tate and generally known as rigid analytic geometry (other methods include the theories of Berkovich spaces and Huber's adic spaces).

Once and for all, fix a field k that is complete for a nonarchimedean valuation $|\cdot|$ (by definition, a valuation here is multiplicative and nontrivial). We can then consider the ring of integers

$$R := \{t \in k : |t| \leq 1\},$$

which is a local ring with maximal ideal

$$\mathfrak{m} := \{t \in k : |t| < 1\},$$

and residue field

$$\tilde{k} := R/\mathfrak{m}.$$

There are three basic examples to keep in mind. First (the "mixed characteristic" case) consider $k = \mathbb{Q}_p$ with its usual p-adic valuation; then $R = \mathbb{Z}_p$, $\mathfrak{m} = p\mathbb{Z}_p$, and $\tilde{k} \simeq \mathbb{F}_p$. Second (the "equicharacteristic" case) consider $k = \mathbb{F}_p((T))$ with valuation given by, say, $|f| = p^{-\operatorname{ord} f}$; then $R = \mathbb{F}_p[[T]]$, $\mathfrak{m} = T\mathbb{F}_p[[T]]$, and again $\tilde{k} \simeq \mathbb{F}_p$. Third (also an equicharacteristic case, although it tends not to come up in number theory), consider $k = \mathbb{R}((T))$ with analogous valuation; the ring of integers, maximal ideal, and residue field are as in the second case except with $\mathbb{R}$ in place of $\mathbb{F}_p$.

We want to define a class of spaces over k. In order to do that, we first define a class of k-algebras (the k-affinoid algebras), we next consider their maximal spectra as a locally ringed space with a given topological structure, and then we glue together these affinoid spaces to form general rigid analytic spaces:

$$\{k\text{-affinoid algebras}\} \rightsquigarrow \{\text{affinoid spaces}\} \rightsquigarrow \{\text{rigid analytic spaces}\}.$$

The analogy to the construction of schemes or complex analytic spaces is clear: in the former case we have, more or less,

$$\{\text{rings}\} \rightsquigarrow \{\text{affine schemes}\} \rightsquigarrow \{\text{schemes}\}$$

and in the latter we have

$$\{\text{set of hol. functions on } U \subset \mathbb{C}^n\} \rightsquigarrow \{\text{vanishing loci of set}\} \rightsquigarrow \{\text{complex analytic spaces}\}.$$

2. The Tate algebra and its geometry

Here's our basic algebraic object, the Tate algebra:

$$T_n(k) := \left\{ \sum_\nu a_\nu X^\nu : a_\nu \in k, |a_\nu| \to 0 \text{ as } \nu \to \infty \right\},$$

where the sum is taken over all n-tuples of nonnegative integers. This is the subring of $k[[X_1, \ldots, X_n]]$ whose coefficients tend to zero with respect to the given valuation on k. We call these power series "strictly convergent" because of the following observation: in any non-archimedean field, we have

$$\sum a_i \text{ converges} \iff |a_i| \to 0 \text{ as } i \to \infty.$$

So for any $f \in k[[X_1, \ldots, X_n]]$,

$$f(b_1, \ldots, b_n) \text{ converges for all } b_i \in R \iff f \in T_n(k).$$

That is, $T_n(k)$ is the set of power series that converge on the unit ball. Since we have fixed a single field k, we will write $T_n = T_n(k)$ for short.

These rings satisfy several very nice properties:

Theorem 2.1. *The Tate algebra T_n is Noetherian, regular, a UFD, Japanese, and Jacobson.* □

Japaneseness is the condition that the its integral closure in any finite extension of the quotient field is itself a finite extension. A ring is Jacobson if every prime ideal is the intersection of the maximal ideals containing it, so in particular any element lying in every maximal ideal is nilpotent (this is the algebraic criterion that ensures that we have some sort of Nullstellensatz).

The proof of this theorem is not deep, but some of it is not easy. Most of the work is in developing an analogue of the Weierstrass preparation theorem in order to be able to induct on dimension.

The Tate algebra also possesses a very nice distinguished norm $||\cdot||$, the Gauss norm, so named because it is analogous to Gauss's content of a polynomial (used in proving Gauss's lemma in basic algebra):

$$\left|\left|\sum_\nu a_\nu X^\nu\right|\right| := \max_\nu |a_\nu|.$$

This is a multiplicative norm (check it!) and T_n is complete for it. It therefore gives T_n the structure of a k-Banach algebra.

In order to give some idea of how these rings work, I'll sketch a proof of another fact about T_n, which is a (perhaps surprising) indication that the algebraic and topological structures cohere nicely:

Lemma 2.2. *Every ideal in T_n is closed.*

Proof. We will prove the more general fact that if A is a k-Banach algebra and M is a normed A-module such that $\hat{M}$ is a finitely generated A-module, then $M = \hat{M}$ (i.e., M is already complete). Applying this to $M = I$ (and using the Noetherianity of T_n) will yield the lemma.

Choose an integer n and a map π such that

$$A^n \xrightarrow{\pi} \hat{M} \to 0$$

is exact. The open mapping (or Banach-Schauder) theorem holds in our case (the proof goes through verbatim), so π is an open map. If we let

$$B^-(A^n) := \left\{(a_1, \ldots, a_n) \in A^n : \max_i\{||a_i||\} < 1\right\}$$

denote the "open unit ball" of A^n, then this implies that $\pi(B^-(A^n))$ is a neighborhood of 0 in $\hat{M}$. Let $x_1, \ldots, x_n$ be the images of the canonical basis of A^n in $\hat{M}$. By the denseness of M in $\hat{M}$, we must have

$$x_i \in M + \pi(B^-(A^n))$$

for each i. Elements of $\pi(B^-(A^n))$ are topologically nilpotent (i.e. their powers tend to zero), so a topological version of Nakayama's lemma immediately implies that $M = \hat{M}$. □

Let's make a step towards geometry by looking at the geometric object associated with T_n. The underlying set of the geometric object will be defined as

$$M(T_n) := \mathrm{MaxSpec}(T_n).$$

Let's find some elements of this set. Let

$$B\left(\overline{k}^n\right) := \left\{(x_1, \ldots, x_n) \in \overline{k}^n : \max_i\{|x_i|\} \leq 1\right\}$$

(the "closed unit ball" in $\overline{k}^n$) and choose any $x \in B\left(\overline{k}\right)$. Let $\mathfrak{m}_x$ be the ideal of all elements of T_n that vanish on x, which makes sense precisely because x lies in the unit ball. I claim that $\mathfrak{m}_x \in M(T_n)$, which follows simply from the observation that $\mathfrak{m}_x$ is the kernel of the natural evaluation map $T_n \to k(x)$ (here we have to use the fact that k is complete, which implies any finite extension of k is complete for a unique extension of the valuation, which implies that $f(x) \in k(x)$ for $f \in T_n$). So we've found an element of $M(T_n)$. It is easy to note that $\mathfrak{m}_x = \mathfrak{m}_y$ if and only if x and y are Galois conjugates, so we have constructed an injective map

$$B\left(\overline{k}^n\right)/\mathrm{Gal}(\overline{k}/k) \xrightarrow{\tau} M(T_n).$$

Proposition 2.3. *The map τ is a bijection. In particular, all elements of $M(T_n)$ arise from the above construction.*

Proof. Obviously it suffices to show that τ is surjective, which is essentially the (weak) Nullstellensatz for T_n: i.e., we want to show that $T_n/\mathfrak{m}$ is a finite extension of k for all $\mathfrak{m} \in M(T_n)$. If we know this, then by picking an embedding $T_n/\mathfrak{m} \to \overline{k}$ we get a point $x = (x_1, \ldots, x_n) \in \overline{k}^n$ by letting x_i be the image (considered in $\overline{k}$) of the coordinate function $X_i \in T_n$ in $T_n/\mathfrak{m}$. It is then trivial that $\mathfrak{m} = \mathfrak{m}_x$. But the weak Nullstellensatz is a formal consequence of T_n being Jacobson. □

So, summing up,

$$M(T_n) = B\left(\overline{k}^n\right)/\mathrm{Gal}(\overline{k}/k)$$

as a set.

Let's point out one more cool thing. Because all of the residue fields $k(x)$ are finite extensions of k, hence complete for a unique extension of the valuation, we can compose the evaluation map with this valuation (also denoted $|\cdot|$) to get, for each $x \in B\left(\overline{k}^n\right)$, a map

$$T_n \to k(x) \to \mathbb{R}_{>0}$$

given by

$$f \mapsto f(x) \mapsto |f(x)|.$$

Furthermore this map is clearly Galois-compatible, so we descend to the quotient and evaluate functions on points of $M(T_n)$, getting positive real numbers. Using this evaluation map, we have the following:

Proposition 2.4 (Maximum modulus principle)**.**

$$\sup_{x \in M(T_n)} |f(x)| = \max_{x \in M(T_n)} |f(x)| = ||f||.$$

Proof. If $f = 0$ this is trivial. Otherwise, without loss of generality, we can scale so that $||f|| = 1$. Consider $\tilde{f}$, the image of f in the residue ring of T_n (by which we mean "elements of Gauss norm ≤ 1 modulo elements of Gauss norm < 1"). It is not difficult to see that this residue ring is canonically $\tilde{k}[X_1, \ldots, X_n]$. As $\overline{\tilde{k}}$ is infinite and $\tilde{f} \neq 0$ identically, there exists some $\tilde{x} \in \overline{\tilde{k}}$ such that $\tilde{f}(\tilde{x}) \neq 0$, which is the same as saying $\widetilde{f(x)} \neq 0$ for any lift x of $\tilde{x}$ to k, which is the same as saying $|f(x)| = 1$. Thus f reaches its maximum at x. □

In some sense, this proposition is telling us that the space $M(T_n)$ is "trying to be compact."

3. Affinoid algebras and their geometry

Now we want to consider more spaces than just $M(T_n)$. Define the category of k-affinoid algebras to have objects

$$\{k\text{-algebras } A \text{ admitting an isomorphism } A \simeq T_n/I \text{ for some } n \text{ and some ideal } I \in T_n\}$$

and morphisms

$$\{k\text{-algebra homomorphisms}\}.$$

Any algebra T_n/I admits a norm via the Gauss norm on T_n, though it may not be multiplicative or even power-multiplicative. This norm yields a k-Banach space structure. If A is any k-affinoid algebra, therefore, it admits a k-Banach space structure, which unfortunately may not be unique. However, the resulting topology is independent of the presentation.

One may protest that the above category does not take the topological structure into account at all and may therefore contain topologically pathological (i.e. discontinuous) morphisms. Fortunately, one can with some difficulty show that all k-algebra homomorphisms are automatically continuous, so no such pathologies occur.

The category of k-affinoid algebras is closed under finite homomorphisms, quotients, direct products, and completed tensor products. All k-affinoid algebras are

Noetherian and Jacobson, all ideals are closed, and we have a Noether normalization theorem that states that all k-affinoid algebras A admit a finite injection $T_d \to A$ where d is the Krull dimension of A.

As before, let

$$M(A) := \mathrm{MaxSpec}(A).$$

If $I = (f_1, \ldots, f_d)$, we should think of $M(T_n/I)$ as the space cut out by setting $f_1 = \ldots = f_d = 0$, just like for varieties over a field. As expected, a map $g : A \to A'$ gives rise to a map $M(g) : M(A') \to M(A)$, so we define the category of k-affinoid spaces to be the opposite category to the category of k-affinoid algebras.

In this more general setting we still have a maximum modulus principle: for any $f \in A$ we have

$$\sup_{x \in M(A)} |f(x)| = \max_{x \in M(A)} |f(x)| < \infty.$$

We take this as the *definition* of the intrinsic seminorm on each k-affinoid algebra (which is a norm if A is reduced), independent of presentation. It is power-multiplicative.

4. Interlude: Why MaxSpec?

Here are a few heuristic reasons why we use MaxSpec instead of Spec to capture the geometry of k-affinoid algebras. *A priori*, of course, one might want to keep the added flexibility that generic points lend to scheme theory.

First, k-affinoid algebras are Jacobson, which implies that a function $f \in A$ that vanishes everywhere on $M(A)$ must be nilpotent. There is no need to consider prime ideals in order for this to be true. This is similar to the situation for finitely generated k-algebras (that is, algebraic varieties) and, indeed, one can certainly study algebraic varieties just using MaxSpec.

Second, as we have seen, it is possible to uniquely evaluate $|f(x)|$ as x ranges over the maximal spectrum of A, but not necessarily over the prime spectrum.

Third, and perhaps most importantly, there are many maps of k-affinoid spaces that do not correspond to any reasonable way of mapping generic points. As an example, consider the natural embedding of the closed ball of radius r, for some $0 < r < 1$ in the image of $|\cdot|$, into the ball of radius 1. Where would the generic points go?

5. Topologize and globalize

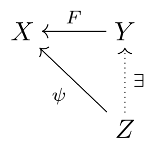
We can endow $M(A)$ with the topology inherited from k, but it will be totally disconnected and therefore almost totally useless. To improve it, we need to somehow "force" certain disconnected subsets to be connected and certain non-compact subsets to be compact. We can do this by restricting both the allowable open sets to the *admissible opens* and (more importantly) restricting the allowable open covers to the *admissible covers*, getting the *strong Tate topology* on $M(A)$.[1]

The framework for doing this is a weak Grothendieck topology, or G-topology. I am not going to formalize this notion. The "weak" refers to the fact that the open sets of the topology are actual sets of points, and the open covers are (some subset of) sets of actual sets whose union is the set in question. In this respect, G-topologies are nicer than, say, the étale site on a scheme. For us, the major

[1] There is also a weak Tate topology, which we will not discuss.

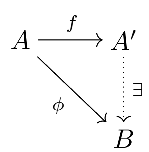
difference between G-topologies and regular topologies will be that the admissible open sets in a G-topology are not necessarily closed under union.

Here is an auxiliary definition. Let X be a k-affinoid space. $U \subseteq X$ is an *affinoid subdomain* if there exists a map of k-affinoid spaces $F : Y \to X$ such that $F(Y) \subseteq U$ and such that for any other map of k-affinoid spaces $\psi : Z \to X$, ψ factors through Y if and only if $\psi(Z) \subseteq U$. In diagrams, we want



if and only if $\psi(Z) \subseteq U.$

In a nutshell, we want U to act like the image of an open immersion. Of course, we can reverse all the arrows in the above definition to state things in terms of k-affinoid algebras, if we so choose: $U \subset M(A)$ is an affinoid subdomain if there exists a map of k-affinoid algebras $f : A \to A'$ such that the image of $M(A)$ under $M(f)$ lies in U and for any map of k-affinoid algebras $\phi : A \to B$, there exists a map $\nu : A' \to B$ such that $\nu \circ f = \phi$ if and only if $M(\phi)$ lands in U. In diagrams,



if and only if $\operatorname{im}(M(\psi)) \subseteq U.$

To me, this way of stating the definition is rather more confusing, but both are obviously equivalent by definition. By the Yoneda lemma, the A' in question is unique up to isomorphism, so denote it as A_U. Clearly, A_U should be thought of as the coordinate ring of U.

By using the completed tensor product on the k-affinoid algebra side, one can prove rather quickly that affinoid subdomains are closed under pullback and intersection.

Now we define the G-topology on a k-affinoid space X. The *admissible open subsets* $U \subseteq X$ are those that have a covering $\{U_i\}$ by affinoid subdomains such that for any $\psi : Y \to X$ such that $\psi(Y) \subseteq U$, the pullback of $\{U_i\}$ along ψ has a finite subcover. A set of subsets $\{V_i\}$ is an *admissible cover* of $V = \cup_i V_i$ if for any $\psi : Y \to X$ such that $\psi(Y) \subseteq V$, the pullback of $\{V_i\}$ along ψ can be refined to a cover by finitely many affinoid subdomains.

In particular, wading through these definitions, we find that our V is itself forced to be admissible, and on the other hand the covering $\{U_i\}$ in the first part of the definition is itself an admissible cover.

We have a set, we have a topology, now we want a structure sheaf. It is a serious theorem that we have enough to get one:

Theorem 5.1 (Tate's acyclicity theorem)**.** *The assignment $U \mapsto A_U$ gives rise to a unique structure sheaf $\mathcal{O}_A$ with respect to the Tate topology. That is, there is a unique way of extending this assignment to all admissible open sets U such that if $\{U_i\}$ is an admissible cover of U, the natural sequence*

$$0 \mapsto \mathcal{O}_A(U) \mapsto \prod \mathcal{O}_A(U_i) \mapsto \prod \mathcal{O}_A(U_i \cap U_j)$$

is exact. □

The proof involves a reduction to a very specific case (a specific simple cover by *rational domains*, which is a type of affinoid subdomain I will not define) and then an explicit calculation. Tate's theorem immediately tells us a few helpful things about the Tate topology. For instance, as T_n is a domain, the rigid analytic unit ball must be connected, which is a good sign.

We define

$$\mathrm{Sp}(A) := (M(A), \mathcal{O}_A).$$

This is the a k-affinoid space, together with its structure sheaf. Finally, a *rigid analytic space* over k is any G-topologized locally ringed space that is locally isomorphic to a k-affinoid space. Morphisms are taken with respect to the category of G-topologized locally ringed spaces. Like for schemes, we can glue rigid analytic spaces in the obvious way.

Here are some things that we can do with rigid analytic spaces. We have a good notion of coherent sheaves and we can calculate the sheaf cohomology of coherent sheaves using Čech cohomology. We have an analytification functor $X \mapsto X^{\mathrm{an}}$ from the category of locally finite type k-schemes to rigid spaces over k, and GAGA holds in this setting (actually, the proof goes through essentially the same way). We can borrow lots of words and definitions from scheme theory, and they act largely as we expect them to: quasicompact, quasiseparated, closed immersion, separated, proper, flat, faithfully flat.

Here are some things that we can't do with rigid analytic spaces. Base change from k to an overfield K is possible for k-affinoids via completed tensor products, and in general for quasiseparated rigid analytic spaces if one is sufficiently careful, but it is not possible in general (and when it is possible, we have to "go outside the category" to do it, unlike in scheme theory where base change is a legitimate morphism). Arguments using stalks are not always possible, as for a G-topologized locally ringed space there are, for example, sheaves that vanish at every stalk but are not the zero sheaf. We have no criterion that corresponds to Serre's criterion for affineness (the statement that a qcqs scheme is affine if and only if its quasicoherent cohomology groups all vanish); more worryingly, we don't even have a good theory of quasicoherent cohomology at all (the naïve definition suffers severe drawbacks). Finally, we can't (or at least, no one has been able to) prove a lot of things that we would like to prove without using some heavier machinery, like Berkovich spaces or input from the theory of formal schemes.

6. Constructing some spaces

Historically, Tate developed the theory of rigid analytic spaces in order to uniformize p-adic objects in much the same way that, for example, $\mathbb{C}$ uniformizes complex elliptic curves over. In this section we'll briefly look at a few important rigid analytic spaces.

To construct the affine space $\mathbb{A}^n$, we glue together balls of bigger and bigger radii. More specifically, let $\mathbb{B}^n = \mathrm{Sp}(T_n)$. Make countably many copies of this unit ball, denoted D_j, indexed by $j \geq 1$, each with coordinates $X_{1,j}, \ldots, X_{n,j}$. Map

$$D_j \to D_{j+1} \qquad \text{via} \qquad X_{i,j+1} \mapsto cX_{i,j} \qquad \text{for each } i,$$

where $c \in k$ is some element such that $0 < |c| < 1$. Glue along these maps. One can check fairly readily that $\mathbb{A}^n$ has the expected universal property of affine n-space.

To construct projective space $\mathbb{P}^n$, we can glue together copies of $\mathbb{A}^n$ in the usual way, or we can eliminate the middleman and do just as well by gluing together copies of $\mathbb{B}^n$ directly.

The p-adic analogue of the upper half plane is the Drinfeld upper half plane. I will not write out a full construction, but it is a spaces whose $\hat{\bar{k}}$-valued points are $\mathbb{P}^1(\hat{\bar{k}}) \setminus \mathbb{P}^1(k)$. The Drinfeld upper half plane admits higher dimensional generalizations as well; the $\hat{\bar{k}}$-valued points are then projective space minus the union of all k-rational hyperplanes.

Given an elliptic curve E over k with $|j(E)| > 1$ (i.e., split multiplicative reduction), Tate showed that there is a unique $q \in k^\times$, $|q| < 1$, such that

$$E^{\mathrm{an}} \simeq \mathbb{G}_m / q^{\mathbb{Z}},$$

and every such q occurs. Here $\mathbb{G}_m$ is the multiplicative group scheme over k, which can be cut out of $\mathbb{A}^2$ in the usual way. Of course, one has to show that the quotient exists as a rigid analytic space. Many generalizations of this construction have been given: Mumford uniformized abelian varieties (with specific reduction), while Cerednik and Drinfeld showed how to uniformize Shimura curves (again with specific reduction). In both cases the uniformizing space is the Drinfeld upper half plane modulo some arithmetic group. Rapoport and Zink have similarly made progress with some more general Shimura varieties, again using the (higher-dimensional version of the) Drinfeld upper half-plane.

7. References

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