POLYNOMIAL-GEOMETRIC SUMS

EVAN WARNER

As Peter Sarnak is fond of saying, the only infinite sum that we really know how to calculate is the geometric sum. Multiplying the summand by a polynomial yields also an explicitly summable result, albeit a combinatorially more difficult one:

Theorem. *Let* $P \in \mathbb{C}[x_1, \dots, x_n]$ *be given by*

$$P(x_1, \dots, x_n) = \sum_{\alpha_1, \dots, \alpha_n = 0}^{N} c_{\alpha_1, \dots, \alpha_n} x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

and $R_1, \dots, R_n \in \mathbb{C}$ *all have absolute value less than one. Then*

$$\sum_{i_1, \dots, i_n = 0}^{\infty} \prod_{\ell=1}^{n} R_\ell^{i_\ell} \cdot P(i_1, \dots, i_n) = \sum_{\alpha_1, \dots, \alpha_n = 0}^{N} c_{\alpha_1, \dots, \alpha_n} \prod_{\ell=1}^{n} \frac{\sum_{i=0}^{\alpha_\ell} A(\alpha_\ell, i-1) R^i}{(1 - R_\ell)^{\alpha_\ell + 1}},$$

where $A(\alpha, k)$ *is the number of permutations of the numbers 1 to* α *in which exactly* k *elements are greater than the previous element (the so-called* Eulerian numbers*), provided we define* $A(\alpha, -1)$ *to be equal to zero if* $\alpha > 0$ *and equal to one if* $\alpha = 0$*.*

Proof. We achieve immediate simplification by linearity, so we only have to deal with one monomial at a time, and in the case of a monomial the multi-parameter sum becomes a product of one-parameter sums (the theorem is written in the above way simply to emphasize the generality). We are left to evaluate sums of the type

$$Q(\alpha) = \sum_{i=0}^{\infty} R^i \cdot i^\alpha,$$

where $|R| < 1$ and α is a nonnegative integer. We have the following formal calculation, whose correctness is justified by the geometric convergence of the R^i term:

$$\frac{\partial Q(\alpha)}{\partial R} = \sum_{i=1}^{\infty} R^{i-1} i^{\alpha+1} = \frac{1}{R} \sum_{i=0}^{\infty} R^i \cdot i^{\alpha+1} = \frac{1}{R} Q(\alpha + 1). \tag{1}$$

Additionally, the ordinary geometric series yields

$$Q(0) = \frac{1}{1 - R}.$$

If we let

$$Q(\alpha) = \frac{T_\alpha(R)}{(1 - R)^{\alpha+1}},$$

then the quotient rule together with (1) gives

$$Q(\alpha + 1) = R \cdot \frac{\partial Q(\alpha)}{\partial R} = R \cdot \left(\frac{T'_\alpha(R) \cdot (1 - R) + (\alpha + 1) T_\alpha(R)}{(1 - R)^{\alpha+2}} \right).$$

Therefore we have the recurrence relation

$$T_{\alpha+1}(R) = R \cdot (T'_\alpha(R) \cdot (1-R) + (\alpha-1)T_\alpha(R)). \tag{2}$$

By induction it is clear that $\deg(T_\alpha) \leq \alpha$. Let

$$T_\alpha(R) = \sum_{i=0}^{\alpha} c_{\alpha,i} R^i.$$

Then (2) yields

$$\begin{aligned}\sum_{i=0}^{\alpha+1} c_{\alpha+1,i} R^i &= R \cdot \left(\sum_{i=0}^{\alpha} (i+1) c_{\alpha,i+1} R^i - \sum_{i=0}^{\alpha} i c_{\alpha,i} R^i + (\alpha+1) \sum_{i=0}^{\alpha} c_{\alpha,i} R^i \right) \\ &= \sum_{i=1}^{\alpha} (i \cdot c_{\alpha,i} + (\alpha - i + 2) \cdot c_{\alpha,i-1}) R^i,\end{aligned}$$

which implies that $c_{\alpha,i}$ satisfies the recurrence relation

$$c_{\alpha+1,i} = i \cdot c_{\alpha,i} + (\alpha - i + 2) \cdot c_{\alpha,i-1} \tag{3}$$

together with the initial value $c_{0,0} = 1$ and $c_{\alpha,0} = 0$ for $\alpha > 0$.

It remains to show that $c_{\alpha,i} = A(\alpha, i-1)$. By assumption, their initial values match. If we know $A(\alpha-1, k-1)$ and $A(\alpha-1, k)$, then we can calculate $A(\alpha, k)$ as follows, arguing combinatorially: all permutations with of length α with precisely k ascents are achieved by inserting α somewhere in a permutation of length $\alpha - 1$ with either $k-1$ or k ascents. In the former case, placing α anywhere except in between an ascent that already exists will increase the number of ascents by one. In the latter case, placing α in any ascent, or at the beginning, will maintain the number of ascents. Therefore

$$A(\alpha, k) = (\alpha - k)A(\alpha-1, k-1) + (k+1)A(\alpha-1, k).$$

This recursion formula, when compared to (3), shows that $c_{\alpha,i} = A(\alpha, i-1)$ and completes the proof. □

The combinatorial definition of the Eulerian numbers immediately implies the relations

$$A(\alpha, k) = A(\alpha, \alpha - k - 1)$$

and

$$\sum_{k=0}^{\alpha-1} A(\alpha, k) = \alpha!$$

for $\alpha > 0$. In particular, the latter relation implies that the coefficients of T_α will grow quite rapidly in general. In addition, one can prove that

$$A(\alpha, k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha+1}{j} (k+1-j)^\alpha,$$

yielding the truly closed-form relation

$$\sum_{i=0}^{\infty} R^i \cdot i^\alpha = \frac{\sum_{k=0}^{\alpha} (\alpha+1)! \left(\sum_{j=0}^{k-1} \frac{(-1)^j}{j!(\alpha+1-j)!} (k-j)^\alpha \right) R^k}{(1-R)^{\alpha+1}}.$$

A calculation using the recurrence relation or the closed-form solution yields the first few values of T_α:

$$\begin{aligned}
T_0(R) &= 1,\\
T_1(R) &= R,\\
T_2(R) &= R^2 + R,\\
T_3(R) &= R^3 + 4R + R,\\
T_4(R) &= R^4 + 11R^3 + 11R^2 + R,\\
T_5(R) &= R^5 + 26R^4 + 66R^3 + 26R^2 + R,\\
T_6(R) &= R^6 + 57R^5 + 302R^4 + 302R^3 + 57R^2 + R,\\
T_7(R) &= R^7 + 120R^6 + 1191R^5 + 2416R^4 + 1191R^3 + 120R^2 + R.
\end{aligned}$$

We can also use the recurrence relation to calculate, for example, $A(\alpha, 1)$ and $A(\alpha, 2)$ for all α. We know that $A(\alpha, 0) = 1$ for all $\alpha > 0$ and $A(\alpha, \alpha) = 1$, so the recurrence relation yields

$$A(\alpha + 1, 1) = 2 \cdot A(\alpha, 1) + \alpha$$

for all $\alpha > 1$. The general method of solving such recurrence relations, which is presented in some detail below in another context, gives us

$$A(\alpha, 1) = 2^\alpha - \alpha - 1.$$

Going one step further, the recurrence relation yields

$$A(\alpha + 1, 2) = 3 \cdot A(\alpha, 2) + (\alpha - 1) \cdot A(\alpha, 1) = 3 \cdot A(\alpha, 2) + (\alpha - 1) \cdot (2^\alpha - \alpha - 1)$$

for $\alpha > 2$, and the same general method (with a fair amount of irritating algebra) gives us

$$A(\alpha, 2) = 3^\alpha - (\alpha + 1)2^\alpha + \frac{1}{2}\alpha(\alpha + 1).$$

We could, of course, continue in this way.

Although there was no real need to consider multi-parameter sums in the above theorem, there are alternative "ad-hoc" methods which, using the techniques of linear recurrence relations, evaluate particular sums without first splitting into monomials. An example follows, which arose in the context of a question in computational number theory:

Theorem. *Let n be a positive integer and $L_1, \ldots, L_n$ be complex numbers such that $|L_\ell| > 1$ for $1 \le \ell \le n$. Then*

$$\sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{n} i_\ell}}{\prod_{\ell=1}^{n} (L_\ell)^{i_\ell}} \left(\prod_{\ell=1}^{n} i_\ell + 1 \right) = \left[\prod_{\ell=1}^{n} \frac{L_\ell}{(1 + L_\ell)^2} \right] \cdot \left[\prod_{\ell=1}^{n} (1 + L_\ell) + (-1)^n \right].$$

Proof. Absolute convergence is obvious, so rearrangements will be made at will. For $1 \le k \le n$, let

$$S_k = \sum_{i_1, i_2, \ldots, i_k = 0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k} i_\ell}}{\prod_{\ell=1}^{k} L_\ell^{i_\ell}} \left(\prod_{\ell=1}^{k} i_\ell + 1 \right). \tag{4}$$

We will essentially work by induction on k, so we first calculate

$$S_1 = \sum_{i=0}^{\infty} \left(-\frac{1}{L_1}\right)^i (i+1) = \frac{L_1^2}{(1+L_1)^2}. \tag{5}$$

By bringing the kth summation to the inside, we get

$$\begin{aligned} S_k &= \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \left[\sum_{i_k=0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k-1} i_\ell} (-1)^{-k}}{\prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} \left(\prod_{\ell=1}^{k} i_\ell + 1 \right) \right] \\ &= \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k-1} i_\ell}}{\prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} \left[\sum_{i_k=0}^{\infty} \left(\frac{-1}{L_k} \right)^{i_k} \left(\left(\prod_{\ell=1}^{k-1} i_\ell \right) \cdot i_k + 1 \right) \right]. \end{aligned}$$

Let

$$D_k = \prod_{\ell=1}^{k-1} i_\ell, \qquad R_k = -\frac{1}{L_k}.$$

Then the sum in brackets becomes

$$\begin{aligned} \sum_{i=0}^{\infty} (D_k \cdot i + 1) R_k^i &= \frac{D_k \cdot R_k}{(1-R_k)^2} + \frac{1}{1-R_k} \\ &= -\frac{L_k}{(1+L_k)^2} (D_k - L_k - 1), \end{aligned}$$

and the whole sum S_k is equal to

$$-\frac{L_k}{(1+L_k)^2} \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k-1} i_\ell}}{\prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} \left[D_k - L_k - 1 \right].$$

In order to derive a relation for S_k in terms of S_{k-1}, we split up the sum into two parts. In the second line, we use (4).

$$\begin{aligned} S_k &= -\frac{L_k}{(1+L_k)^2} \left[\sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k-1} i_\ell}}{\prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} (D_k + 1) - \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k-1} i_\ell}}{\prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} (L_k + 2) \right] \\ &= -\frac{L_k}{(1+L_k)^2} \left[S_{k-1} - (L_k+2) \prod_{\ell=1}^{k-1} \left(\sum_{i_\ell=0}^{\infty} \left(-\frac{1}{L_k} \right)^{i_\ell} \right) \right] \\ &= -\frac{L_k}{(1+L_k)^2} \left[S_{k-1} - (L_k+2) \prod_{\ell=1}^{k-1} \frac{L_\ell}{1+L_\ell} \right]. \end{aligned}$$

This is a first-order inhomogeneous recurrence relation. We solve it as follows: let

$$A_k = -\frac{L_{k+1}}{(1+L_{k+1})^2}, \qquad B_k = \frac{L_{k+1}(L_{k+1}+2)}{(1+L_{k+1})^2} \prod_{\ell=1}^{k} \frac{L_\ell}{1+L_\ell}, \tag{6}$$

so

$$S_{k+1} = A_k S_k + B_k. \tag{7}$$

As $|L_\ell| > 1$ for each ℓ, we see that $A_\ell \neq 0$ for each ℓ, so upon dividing (7) by $\prod_{\ell=1}^{k} A_\ell$, we get

$$\frac{S_{k+1}}{\prod_{\ell=1}^{k} A_\ell} - \frac{S_k}{\prod_{\ell=1}^{k-1} A_\ell} = \frac{B_k}{\prod_{\ell=1}^{k} A_\ell}.$$

Note that if $k=1$, then $\prod_{\ell=1}^{k-1} A_\ell$ is the empty sum; i.e., equal to 1. Now sum up these equations from $k=1$ to $k=n-1$. This sum telescopes, and we are left with

$$\frac{S_n}{\prod_{\ell=1}^{n-1} A_\ell} - S_1 = \sum_{k=1}^{n-1} \frac{B_k}{\prod_{\ell=1}^{k} A_\ell},$$

or upon rearrangement,

$$S_n = \left[\prod_{\ell=1}^{n-1} A_\ell\right] \cdot \left[S_1 + \sum_{k=1}^{n-1} \frac{B_k}{\prod_{\ell=1}^{k} A_\ell}\right].$$

Upon plugging in (5) and (6),

$$\begin{aligned}
S_n &= \left[\prod_{\ell=1}^{n-1}\left(-\frac{L_{\ell+1}}{(1+L_{\ell+1})^2}\right)\right] \cdot \left[\frac{L_1^2}{(1+L_1)^2} + \sum_{k=1}^{n-1}\left(\frac{\frac{L_{k+1}(L_{k+1}+2)}{(1+L_{k+1})^2}\prod_{\ell=1}^{k}\frac{L_\ell}{1+L_\ell}}{\prod_{\ell=1}^{k}\left(-\frac{L_{\ell+1}}{(1+L_{\ell+1})^2}\right)}\right)\right] \\
&= \left[(-1)^{n-1}\prod_{\ell=1}^{n-1}\frac{L_{\ell+1}}{(1+L_{\ell+1})^2}\right] \cdot \left[\frac{L_1^2}{(1+L_1)^2} + \sum_{k=1}^{n-1}(-1)^k(L_{k+1}+2)\frac{\prod_{\ell=1}^{k}\frac{L_\ell}{1+L_\ell}}{\prod_{\ell=2}^{k}\frac{L_\ell}{(1+L_\ell)^2}}\right] \\
&= \left[(-1)^{n-1}\prod_{\ell=2}^{n}\frac{L_\ell}{(1+L_\ell)^2}\right] \cdot \left[\frac{L_1^2}{(1+L_1)^2} + \frac{L_1}{1+L_1}\sum_{k=1}^{n-1}(-1)^k(L_{k+1}+2)\prod_{\ell=2}^{k}(1+L_\ell)\right] \\
&= \left[(-1)^{n-1}\prod_{\ell=1}^{n}\frac{L_\ell}{(1+L_\ell)^2}\right] \cdot \left[L_1 + \sum_{k=1}^{n-1}(-1)^k(L_{k+1}+2)\prod_{\ell=1}^{k}(1+L_\ell)\right].
\end{aligned}$$

Consider the expression in the second pair of straight brackets. By writing $L_{k+1} + 2 = (L_{k+1}+1)+1$, we see that it is equal to

$$\begin{aligned}
L_1 + \sum_{k=1}^{n-1}&\left[(-1)^k(L_{k+1}+1)\prod_{\ell=1}^{k}(1+L_\ell) + (-1)^k\prod_{\ell=1}^{k}(1+L_\ell)\right] \\
&= L_1 + \sum_{k=1}^{n-1}\left[(-1)^k\prod_{\ell=1}^{k+1}(1+L_\ell) + (-1)^k\prod_{\ell=1}^{k}(1+L_\ell)\right].
\end{aligned}$$

This sum clearly telescopes due to the alternating signs $(-1)^k$, and we are left with

$$\begin{aligned}
L_1 + (-1)^{n-1}\prod_{\ell=1}^{n}(1+L_\ell) &+ (-1)^1(1+L_1) \\
&= (-1)^{n-1}\prod_{\ell=1}^{n}(1+L_\ell) - 1.
\end{aligned}$$

Plugging this back into our expression for S_n yields

$$\begin{aligned}
S_n &= \left[(-1)^{n-1}\prod_{\ell=1}^{n}\frac{L_\ell}{(1+L_\ell)^2}\right] \cdot \left[(-1)^{n-1}\prod_{\ell=1}^{n}(1+L_\ell) - 1\right] \\
&= \left[\prod_{\ell=1}^{n}\frac{L_\ell}{(1+L_\ell)^2}\right] \cdot \left[\prod_{\ell=1}^{n}(1+L_\ell) + (-1)^n\right],
\end{aligned}$$

which is the desired expression. □