# GROSS-ZAGIER ON SINGULAR MODULI: THE ANALYTIC PROOF

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## 1. Introduction

The famous results of Gross and Zagier compare the heights of Heegner points on modular curves with special values of the derivatives of related $L$-functions. When specialized to the level 1 case (i.e., the full modular curve $\mathbb{H}/\Gamma$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$), we recover an astounding formula for the differences of singular moduli (the Heegner points on the full modular curve) in terms of an explicit prime factorization. The goal of this talk is to sketch an analytic proof of this formula, following Gross and Zagier's paper in Crelle's Journal.

First, the formula. Let $\tau$ lie in an imaginary quadratic extension $K$ of $\mathbb{Q}$. To each $\tau$, we associate a discriminant $d$ in the usual way: if $a\tau^2 + b\tau + c = 0$ and $(a, b, c) = 1$, then $d = b^2 - 4ac$. By the theory of complex multiplication, $j(\tau)$ is an *algebraic integer* in an abelian extension of $K$, of degree $h$ over $\mathbb{Q}$, where $h$ is the class number of primitive binary quadratic forms of discriminant $d$[1] or alternatively the class number of the imaginary quadratic field of discriminant $d$. Its Galois conjugates are the numbers $j(\tau')$, where $\tau'$ runs over the roots of primitive quadratic polynomials of discriminant $d$.

Let $d_1$ and $d_2$ be two relatively prime negative fundamental discriminants (i.e., integers that are either 1 or the discriminant of a quadratic number field). Let $D = d_1 d_2$ and let $w_1$ and $w_2$ be the number of roots of unity in the quadratic orders of discriminant $d_1$ and $d_2$, respectively. Define

$$J(d_1, d_2) = \left( \prod_{[\tau_1],[\tau_2]} (j(\tau_1) - j(\tau_2)) \right)^{\frac{4}{w_1 w_2}},$$

where the product is taken over all equivalence classes modulo $\Gamma$ of elements $\tau_i$ such that the discriminant of $\tau_i$ is equal to $d_i$. The exponent is merely an annoying edge-case fudge factor: if $d_1$ and $d_2$ are less than $-4$, then $w_1 = w_2 = 2$, and by the above characterization of the Galois conjugates of $j(\tau)$ we find therefore

$$J(d_1, d_2) = \mathrm{N}(j(\tau_1) - j(\tau_2)) \quad \text{if } d_1, d_2 < -4,$$

where N is the absolute norm. In particular, in this case $J(d_1, d_2)$ is a rational integer. In all cases, $J^2$ is a rational integer.

For any prime $\ell$ such that $\left(\frac{D}{\ell}\right) \neq -1$, we let $\epsilon(\ell)$ equal $\left(\frac{d_1}{\ell}\right)$ if $\ell$ and $d_1$ are coprime and $\left(\frac{d_2}{\ell}\right)$ if $\ell$ and $d_2$ are coprime. This makes sense because $d_1$ and $d_2$ are

[1]this class number is defined as the number of $\Gamma$-inequivalent forms of a given discriminant, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ has its usual action on binary quadratic forms by change of variables.

coprime, so one of the two options must hold, and if *both* hold then $\left(\frac{d_1}{\ell}\right)$ and $\left(\frac{d_2}{\ell}\right)$ are not equal to zero and

$$\left(\frac{d_1}{\ell}\right)\left(\frac{d_2}{\ell}\right) = \left(\frac{D}{\ell}\right) \neq -1,$$

so they must agree (either as 1 or $-1$). We extend the definition of $\epsilon$ by multiplicativity to all numbers whose prime factors $p$ satisfy $\left(\frac{D}{p}\right) \neq -1$.

**Theorem 1.1** (Gross-Zagier formula)**.**

$$J(d_1, d_2)^2 = \pm \prod_{\substack{x^2 < D \\ x^2 = D \mod 4}} F\left(\frac{D - x^2}{4}\right),$$

*where*

$$F(m) = \prod_{\substack{nn' = m \\ n, n' > 0}} n^{-\epsilon(n)}.$$

*The positive sign holds so long as neither $d_1$ or $d_2$ is equal to $-4$.*

We note two things: first, this makes sense, because $\epsilon$ is getting evaluated only at a number dividing $\frac{D-x^2}{4}$, and any prime $\ell$ dividing that quantity must satisfy $\left(\frac{D}{\ell}\right) \neq -1$. Second, note (exercise!) that $F(m)$ can only be a power of a single prime (the unique prime $\ell$ dividing $m$ to odd exponent such that $\epsilon(\ell) = -1$). Therefore this formula gives us the prime factorization of $J(d_1, d_2)^2$ and shows us that the prime factors cannot be too big (because they must divide something of the form $\frac{D-x^2}{4}$).

As an example, take $d_1 = -67$ and $d_2 = -163$, the last two negative discriminants with class number one, for simplicity's sake. Then, after trivial calculations, we get

$$J(-67, -163) = j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) = \pm 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.$$

One should also mention here that the individual $j$-values here are cubes,[2] so we have here an example of two highly divisible numbers whose difference is also highly divisible. This phenomenon (of "high quality *abc* conjecture triples" showing up) has actually been used in reverse: Granville and Stark proved lower bounds on class numbers of imaginary quadratic fields that were strong enough to make Siegel zeroes impossible, conditional on the *abc* conjecture over arbitrary number fields.

I will not discuss (further) applications of the Gross-Zagier formula for lack of time, and also because Zeb should be giving a talk on this subject soon anyway. Instead, I'll sketch a proof, one of two given in the original paper in Crelle's Journal. Because this is the analytic number theory student seminar, we'll go through the more analytic of the two.

[2]The theorem here is due to Weber: if the class number is 1 and 3 does not divide the discriminant $d$, then the $j$-invariant turns out to be a perfect cube. See, for example, Cox, Primes of the form $x^2 + ny^2$, Theorem 12.2.

## 2. Siegel's idea: the diagonal of an Eisenstein series

The first order of business is to massage the right hand side of the Gross-Zagier formula into a form that is more amenable to algebraic number theory. Set

$$S = \sum_{\substack{x^2<D \\ x^2=D \mod 4}} \sum_{n \mid \frac{x^2-D}{4}} \epsilon(n)\log n.$$

Then upon taking absolute values, logarithms, and a minus sign of both sides, the Gross-Zagier formula up to sign is

$$-\log|J(d_1,d_2)|^2 \stackrel{?}{=} S.$$

We can determine the correct sign if $d_1$ and $d_2$ are less than $-4$ immediately by the aforementioned remark that in this case $J(d_1,d_2)$ is a rational integer, so its square is positive, and the right-hand side of Gross-Zagier is manifestly positive. In the case that one of $d_1$ or $d_2$ is equal to $-3$, the positivity of the sign requires a small separate argument that I will omit.

In this section, we will go a long way towards calculating $S$. Let $K=\mathbb{Q}(\sqrt{D})$, and let $\chi$ be a *genus character* (that is, a quadratic character from the narrow class group of $K$) specified as follows: $\chi(\mathfrak{p})=1$ if $\mathfrak{p}$ is inert, while $\chi(\mathfrak{p})=\epsilon(\mathrm{N}\,\mathfrak{p})$ otherwise. Let $\mathfrak{d}=(\sqrt{D})$ denote the different of $K$.

**Proposition 2.1.**

$$S = \sum_{\substack{\nu\in\mathfrak{d}^{-1} \\ \nu\gg 0 \\ \mathrm{Tr}(\nu)=1}} \sum_{\mathfrak{n}\mid(\nu)\mathfrak{d}} \chi(\mathfrak{n})\log \mathrm{N}(\mathfrak{n}).$$

*Proof.* We set $\nu=\frac{x+\sqrt{D}}{2\sqrt{D}}$ and check that the correspondence $\mathfrak{n}\to\mathrm{N}(\mathfrak{n})$ gives a bijection between ideal divisors of $(\nu\sqrt{D})$ and positive divisors of $\frac{D-x^2}{4}$. Furthermore, with respect to this bijection, $\chi(\mathfrak{n})=\epsilon(\mathrm{N}(\mathfrak{n}))$. □

Here Gross and Zagier take inspiration from Siegel, who wrote down a two-variable Hecke-Eisenstein series corresponding to $K$, restricted to the diagonal, and got a modular form on the full modular group which could then be related to the standard Eisenstein series $E_{2k}$, yielding the formula

$$30k\zeta_K(-k+1) = \sum_{\substack{\nu\in\mathfrak{d}^{-1} \\ \nu\gg 0 \\ \mathrm{Tr}(\nu)=1}} \sum_{\mathfrak{n}\mid(\nu)\mathfrak{d}} \mathrm{N}(\mathfrak{n})^{k-1},$$

valid for $k=2$ and $k=4$. Here, we want a logarithm instead of a power and we need to twist by $\chi$. Twisting is easy enough; to get the logarithm, we note that

$$\left[\frac{d}{ds}n^s\right]_{s=0} = [n^s\log n]_{s=0} = \log n,$$

so perhaps we should be thinking about taking derivatives at zero of the exponent. With this in mind, the strategy is as follows: we write down a (twisted) Eisenstein series $E_s(z,z')$ corresponding to $K$, take its Fourier expansion, take its derivative at $s=0$, and then restrict to the diagonal. We get a function $F(z)$ on $\mathbb{H}$ that transforms like a modular function but satisfies a certain logarithmic growth condition. The piece that we want (i.e., the left hand side of the formula in the above proposition) falls out from the first Fourier coefficient of $F(z)$. We express this coefficient in terms of the constants of growth of the function and do some irritating algebra to get the following:

**Proposition 2.2** (Expanding the right hand side)**.**

$$S = \lim_{s\to 1}\left[2\sum_{\substack{n>\sqrt{D}\\ n=D \mod 2}}\left[\sum_{d\mid\frac{n^2-D}{4}}\epsilon(d)\right]Q_{s-1}\left(\frac{n}{\sqrt{D}}\right) - \frac{4\pi}{\zeta(2s)}\left(h_2'\left|\frac{d_1}{4}\right|^{s/2}\zeta_{K_1}(s)+\right.\right.$$
$$\left.\left. h_1'\left|\frac{d_2}{4}\right|^{s/2}\zeta_{K_2}(s) - h_1'h_2'\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1)\right)\right] - 24h_1'h_2',$$

*where*

$$Q_{s-1}(t) = \int_0^\infty (t+\sqrt{t^2-1}\cosh v)^{-s}\, dv$$

*is the Legendre function of the second kind,* $K_j = \mathbb{Q}(\sqrt{d_j})$*, and*

$$h_j' = \frac{2h_j}{w_j}.$$

*Sketch.* We start by defining the Eisenstein series in question:

$$E_s(z,z') := \sum_{[\mathfrak{a}]\in C_K}\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^{1+2s}\sum_{\substack{(m,n)\in\mathfrak{a}^2/\mathcal{O}^\times\\ (m,n)\neq(0,0)}}\frac{y^2y'^s}{(mz+n)(m'z'n')|mz+n|^{2s}|m'z'+n'|^{2s}},$$

a non-holomorphic Eisenstein series of weight 1 on $\mathrm{SL}_2(\mathcal{O}_K)$ (here $y$ and $y'$ are the imaginary parts of $z$ and $z'$, respectively). Via standard methods invented by Hecke, we can develop a Fourier expansion of $E_s(z,z')$ whose general term involves a suitably generalized "sum of divisors" function. Using the Fourier expansion, we can prove easily that $E_s(z,z')$ has a holomorphic continuation to all $s\in\mathbb{C}$ and that $E_0(z,z')$ vanishes identically, so we can calculate precisely an expression of the form

$$\begin{aligned}F(z) :&= \frac{\sqrt{D}}{8\pi^2}\frac{\partial}{\partial s}E_s(z,z)|_{s=0}\\ &= 4L_K(1,\chi)\log(y) + 4\left[L_K'(1,\chi)+\left(\frac{1}{2}\log D - \log\pi - \gamma\right)L_K(1,\chi)\right]\\ &\quad + \text{[explicit, quickly-converging terms involving sum-of-divisors functions]}.\end{aligned}$$

By construction, this function transforms under $\Gamma$ like a modular form of weight 2, and by the above calculation the function has asymptotic growth like $A\log y + B + O(y^{-\epsilon})$ as $y\to\infty$. If this function were actually a modular form, we could calculate which modular form it was by calculating the first few Fourier coefficients and matching with a linear combination of, say, Eisenstein series. Here we can't do quite as well, but there is a nice lemma (technique due to Sturm) that allows us to calculate an integral over the first Fourier term $a_1(y)$ in terms of $A$ and $B$:

$$\lim_{s\to 0}\left(4\pi\int_0^\infty a_1(y)e^{-4\pi y}y^s\, dy + \frac{24A}{s}\right) = 24A\left(2\frac{\zeta'(2)}{\zeta(2)}+1+\log 4\right) - 24B.$$

Writing $a_1(y)$ explicitly in our case, we have

$$a_1(y) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}(\nu)=1}} \sigma'_\chi((v)\mathfrak{d}) - \sum_{\substack{\nu \in \delta^{-1} \\ \nu > 0 > \nu' \\ \mathrm{Tr}(\nu)=1}} \sigma_{0,\chi}((\nu)\mathfrak{d})\Phi(|\nu'|y),$$

where $\sigma'_\chi = \sum_{\mathfrak{n}|\mathfrak{a}} \chi(\mathfrak{n}) \log \mathrm{N}(\mathfrak{n})$ is the derivative of a sum-of-divisors function (so this is the term that we really want!), $\sigma_{0,\chi}$ is a different sum-of-divisors function, and $\Phi$ is an explicit function given as the derivative at zero of a particular contour integral. Fortunately, the term we want does not depend on $y$, so it pulls out of the integral. The second sum of divisors function (also independent of $y$), when demystify a bit by writing $\nu = \frac{n+\sqrt{D}}{2\sqrt{D}}$, turns out to be equal to

$$\sigma_{0,\chi}((\nu)\mathfrak{d}) = \sum_{d | \frac{n^2-D}{4}} \epsilon(d).$$

Applying the above lemma of Sturm, we get an explicit expression for $S$ in terms of the above sum, a bunch of $L$-values, and a particular messy but computable integral over $\Phi$. Computing it by contour integration, we find it is essentially the Legendre function $Q_{s-1}$, and we end up with

$$S = \lim_{s\to 1} \left[ 2 \sum_{\substack{n > \sqrt{D} \\ n = D \mod 2}} \left[ \sum_{d | \frac{n^2-D}{4}} \epsilon(d) \right] Q_{s-1}\left(\frac{n}{\sqrt{D}}\right) - \frac{\sqrt{D}}{\pi^2} \frac{L_K(1,\chi)}{s-1} \right]$$
$$+ \frac{12\sqrt{D}}{\pi^2} L_K(1,\chi) \left[ 2 + 2\frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2}\log D \right] - \frac{12\sqrt{D}}{\pi^2} L'_K(1,\chi).$$

We then use Taylor expansions of zeta and $L$-functions and the evaluation

$$L(1,\chi_j) = \frac{\pi}{\sqrt{|d_j|}} h'_j$$

(recalling that the regulator of an imaginary quadratic field is exactly 1) to conclude. □

## 3. A Green's function

To approach the left-hand side of the Gross-Zagier formula, we want to construct an automorphic Green's function for the modular group $\Gamma$. By "automorphic," of course, we mean invariant under $\Gamma$. By Green's function, we mean a function that is the kernel of an integral operator that inverts a given differential operator. In this case, our given operator is $\Delta - s(s-1)$, where $\Delta$ is the hyperbolic Laplacian on $\mathbb{H}$.

In this case, the Green's function for this operator on the upper half plane is given explicitly by

$$g_s(\tau_1,\tau_2) = -2Q_{s-1}(\cosh d(\tau_1,\tau_2)) = -2Q_{s-1}\left(\frac{(u_1-u_2)^2 + v_1^2 + v_2^2}{2v_1v_2}\right),$$

where $d$ is the hyperbolic distance and $\tau_i = u_i + iv_i \in \mathbb{H}$. To make it automorphic with respect to $\Gamma$, we just average over $\Gamma$:

$$G_s(\tau_1,\tau_2) := \sum_{\gamma\in\Gamma} g_s(\tau_1, \gamma\tau_2).$$

Consulting Iwaniec's book on the spectral theory of automorphic forms, we find to our relief that there is a whole chapter devoted to the study of these objects. In particular, we know that they are real-analytic off the diagonal, they blow up like $\log|\tau_1-\tau_2|^2+O(1)$ as $\tau_1\to\tau_2$ in $\mathbb{H}/\Gamma$, and we can explicitly calculate their Fourier series (which, in particular, gives us their meromorphic continuation in $s$). Let

$$E(\tau,s)=\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\frac{v^s}{|c\tau+d|^{2s}}$$

be the standard real-analytic Eisenstein series and let

$$\phi(s)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

**Proposition 3.1.** *If $\tau_1,\tau_2\in\mathbb{H}$ are not equivalent under the action of $\Gamma$, then*

$$\log|j(\tau_1)-j(\tau_2)|^2=\lim_{s\to1}(G_s(\tau_1,\tau_2)+4\pi E(\tau_1,s)+4\pi E(\tau_2,s)-4\pi\phi(s))-24.$$

The reason we have to give this expression in terms of a limit as $s\to1$ is because all four functions have a pole there; fortunately, the poles are all simple and the residues cancel out precisely, as can easily be shown using the Fourier expansion.

*Sketch.* This expression may look totally out of left field, and to some extent it is, but the basic principle is the following: two harmonic functions that differ by $o(1)$ at infinity must actually be equal. Consider $\tau_2$ as fixed and look at varying $\tau_1$. The right hand side is harmonic because it is the real part of the analytic function $\log(j(\tau_1)-j(\tau_2))^2$ (remember, $\tau_1$ is not $\Gamma$-equivalent to $\tau_2$!). The left-hand side is harmonic because the first two terms are a limit of eigenfunctions of the Laplacian with the same eigenvalues which are tending to zero (because $s(s-1)\to0$ as $s\to1$) and the second two terms are constant in $\tau_1$. To look at their asymptotic behavior, we use the $q$-expansion of the $j$-function to see that (remember, $\tau_2$ is constant!)

$$\log|j(\tau_1)-j(\tau_2)|^2=\log|e^{-2\pi i\tau_1}+O(1)|^2=4\pi v_1+O(e^{-2\pi v_1})$$

as $v_1\to\infty$. Meanwhile, the left-hand side has been cooked up precisely so that if we look at the Fourier expansions of the terms we find that the asymptotic behavior as $v_1\to\infty$ is also $O(e^{-2\pi v_1})$. □

So far, we have not used that $\tau_1$ and $\tau_2$ are supposed to be imaginary quadratic numbers. We will rectify this now by evaluating all four terms of this sum. The last term, $\phi(s)$, is an explicit function that doesn't depend on $\tau_1$ or $\tau_2$ at all, so we leave well enough alone.

The second and third terms are Eisenstein series which can be explicitly evaluated at imaginary quadratic points when we notice that $|c\tau+d|^2$ is a norm form of an imaginary quadratic number field evaluated on a single ideal class, so we can calculate

$$E(\tau_j,s)=\frac{w_j}{2}\left|\frac{d_j}{4}\right|^{-\frac{s}{2}}\zeta(2s)^{-1}\zeta_{K_j,\mathcal{A}_j}(s),$$

where $\mathcal{A}_j$ is the ideal class in $K_j$ that corresponds to the class of $\tau_j$, so $\zeta_{K_j,\mathcal{A}_j}$ is the partial zeta function calculated by summing over $\mathrm{N}(\mathfrak{a})^{-s}$ for all $\mathfrak{a}\in\mathcal{A}_j$. These partial zeta functions are perhaps not so easy to deal with, but fortunately we are interested in the quantity $\log|J(d_1,d_2)|^2$, which involves a sum over every ideal class. Therefore at the end we get two terms with explicit constants, as above, and full zeta-functions associated to $K_1$ and $K_2$.

The first term is the trickiest. Unfolding the sum that defines $G_s(\tau_1, \tau_2)$, we find that

$$\frac{2}{w_1}\frac{2}{w_2}\sum_{[\tau_1],[\tau_2]} G_s(\tau_1,\tau_2) = \sum_{(\tau_1,\tau_2 \in \mathbb{H}/\Gamma} g_s(\tau_1,\tau_2),$$

where both sums range over $\tau_j$ with discriminant exactly $d_j$. Now we evaluate $g_s$ at imaginary quadratic points. This is tricky but it is *a priori* plausible that we get something calculable because $g_s$ was originally written as a Legendre function evaluated at a rational function of degree two in the real and imaginary parts of $\tau_1$ and $\tau_2$. In fact, if $\tau_j$ satisfies $a_j\tau_j^2 + b_j\tau_j + c_j = 0$, then we set

$$n = 2a_1c_2 + 2a_2c_1 - b_1b_2$$

and find that

$$g_s(\tau_1,\tau_2) = -2Q_{s-1}\left(\frac{n}{\sqrt{D}}\right).$$

Of course, we have to be very careful about correctly counting the $n$. In all we get for the first term

$$-2\sum_{\substack{n>\sqrt{D}\\ n=D \mod 2}} \rho(n)Q_{s-1}\left(\frac{n}{\sqrt{D}}\right)$$

where $\rho(n)$ is the solution to a counting problem: it is equal to half of the number of pairs of $\Gamma$-inequivalent binary quadratic forms whose discriminants are $d_j$ (respectively) and who satisfy the bilinear form $2a_1c_2 + 2a_2c_1 - b_1b_2 = n$ (the factor of a half comes in because we only want to count positive definite forms, discarding their negative definite pairs). If we like, we can put these two conditions together: $\rho(n)$ is the number of $\Gamma$-inequivalent representations of the binary quadratic form

$$d_1x^2 - 2nxy + d_2y^2$$

by the discriminant form.

Putting this all together in one formula, we find that

$$-\log|J(d_1,d_2)|^2 = \lim_{s\to 1}\left[2\sum_{\substack{n>\sqrt{D}\\ n=D \mod 2}} \rho(n)Q_{s-1}\left(\frac{n}{\sqrt{D}}\right) - \frac{4\pi}{\zeta(2s)}\left(h_2'\left|\frac{d_1}{4}\right|^{s/2}\zeta_{K_1}(s) + h_1'\left|\frac{d_2}{4}\right|^{s/2}\zeta_{K_2}(s) - h_1'h_2'\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1)\right)\right] - 24h_1'h_2'.$$

## 4. Evaluation of $\rho(n)$ and conclusion

Comparing the results of expanding the right- and left-hand sides, we immediately see that the remainder of the proof boils down to the proposition

$$\rho(n) \stackrel{?}{=} \sum_{d \mid \frac{n^2-D}{4}} \epsilon(d).$$

In other words, we want to evaluate the number of orbits of an infinite set by an infinite group as a finite sum. The proof is algebraic and fairly irritating; it proceeds by translating everything into the setting of algebraic number theory by interpreting the quadratic forms as norm forms from $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ to $K = \mathbb{Q}(\sqrt{D})$.

Call the following Data 1: triples $[\alpha_1] \in C_{K_1}, [\alpha_2] \in C_{K_2}$, and $\rho \in \alpha_1\alpha_2/U_{K_1}U_{K_2}$ such that

$$\mathrm{N}_{L/K}(\rho) = \mathrm{N}(\mathfrak{a}_1)\,\mathrm{N}(\mathfrak{a}_2)\mu,$$

where $\mu = \frac{n-\sqrt{D}}{2} \in \mathcal{O}_K$ and $U_{K_j}$ is the unit group of $K_j$. Call the following Data 2: integral ideals $\mathfrak{A}$ of $L$ with norm (down to $K$) equal to $(\mu)$. Given an element of Data 1, construct an element of Data 2 by the formula $\mathfrak{A} = \rho^{-1}\mathfrak{a}_1\mathfrak{a}_2$. The claim, which can be verified by class field theory in constructing an inverse map, is that this map is a bijection.

As $\rho(n)$ boils down to counting Data 2 and the finite sum boils down to counting Data 1, this establishes the desired equality.

## 5. References


Cox, D.A., *Primes of the form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication*, Wiley (1997).

Granville, Andrew, and Stark, H.M., *ABC implies no "Siegel zeros" for L-functions of characters with negative discriminant*, Inventiones Math 139 (2000), 509-523.

Gross, B.H., and Zagier, Don B., *On singular moduli*, Journal für die reine und angewandte Mathematik 355 (1984), 191-220.

Iwaniec, H., *Spectral Methods of Automorphic Forms*, second edition, Graduate Studies in Mathematics vol. 53, AMS (2002).