

THREE APPROACHES TO CHOW'S THEOREM

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1. STATEMENT AND OVERVIEW

The focus of this talk will be on Chow's theorem, a simple but very useful result linking complex analytic geometry to algebraic geometry. I'll state the theorem shortly, and give sketches of three increasingly sophisticated methods to prove it: the original proof from 1949 due to Chow, an analytic simplification from 1953 due to Remmert and Stein, and finally as following from the famous GAGA result of Serre in 1956.¹ The fact that these dates all lie in a seven-year span should not imply that the methods are at all similar, in fact, we'll see that the approaches are quite distinct.

We'll start from very basic definitions, in case people haven't seen analytic spaces before. Let \mathcal{H}_n denote the sheaf of holomorphic functions on \mathbb{C}^n .

An *analytic subspace of \mathbb{C}^n* is a ringed space (X, \mathcal{H}_X) such that $X \subseteq \mathbb{C}^n$ is given the Euclidean topology and is locally cut out by holomorphic functions; i.e., for every point $z \in X$ there is a neighborhood $U_z \subseteq \mathbb{C}^n$ such that $X \cap U_z$ is the vanishing locus of finitely many holomorphic functions. We set

$$\mathcal{H}_X = \mathcal{H}_n / \mathcal{A}_X,$$

where \mathcal{A}_X is the ideal sheaf of X ; i.e., locally those holomorphic functions vanishing on X . This is a locally ringed space, merely because $(\mathbb{C}^n, \mathcal{H}_n)$ is and we're just taking quotients. We call X an *analytic subvariety of \mathbb{C}^n* if it is closed as a subset.

An *analytic space* is a Hausdorff² locally ringed space (X, \mathcal{H}_X) that is locally isomorphic to an analytic subvariety of some domain in \mathbb{C}^n .

We say that a subspace Y of an analytic space X is an *analytic subspace* if it is locally an analytic subspace of X (using a chart on X to define what this means) and the structure sheaf on Y is induced from X in the obvious way (taking a quotient by the vanishing ideal). The underlying set of an analytic subvariety is called an *analytic set*. If the underlying analytic set is closed, we say that X is an *analytic subvariety*. We will be concerned almost exclusively with analytic subvarieties of projective space \mathbb{P}^n .

In some sense, analytic spaces are much nicer than schemes, for example because rings can be very much more complicated than rings of holomorphic functions (our local model is algebraically simpler). But in analytic geometry one wants to be able to refer to notions in Euclidean topology, which is of course much finer than the "analytic Zariski topology," and this makes things rather more difficult: the local-to-global aspects of the theory are much trickier.

¹I don't know when this result was first conjectured; Chow offers only that it is "classical."

²There are things that can be said without this assumption, but if Serre restricts to this case I'm happy doing so as well.

Theorem 1.1 (Chow’s theorem). *Every closed analytic subset of \mathbb{P}^n is an algebraic set.*

I would contend that this theorem is manifestly interesting in its own right, but it can also be very useful. If there’s time at the end, I’ll mention a couple of applications.

2. VIA CHOW

Chow’s original proof was long and complicated. The proof is in two steps, which when put together immediately imply the result. I will sketch a proof of the first step, and for the second step only describe an analogy to the theory of diophantine approximation.

Define an *analytic simplex* to be a topological simplex in \mathbb{P}^n that is biholomorphic with the standard simplex in some complex affine space. The dimension of an analytic simplex is the dimension of the standard simplex in question. One should think of an analytic simplex as an element of a triangulation of an analytic set; it is an easy theorem that every analytic set in projective space can be so triangulated. It makes sense to talk about usual topological notions as if we were doing simplicial homology: an *analytic chain* is a formal linear combination of analytic simplexes, and an *analytic cycle* is an analytic chain mapping to zero under the boundary operator. It has to be shown (but is not difficult to show) that an analytic cycle is actually a formal linear combination of closed analytic sets, which we can take to be irreducible. Picking an orientation (ordered list of basis vectors) of the ambient space, all analytic chains have an induced orientation, and thus we can define a topological intersection multiplicity of these chains at an intersection. By a *regular intersection*, we mean that the intersection of the tangent planes at each point of intersection has the “expected” number of dimensions and no more; i.e., the intersection is generic in a suitable sense.

Topologically, \mathbb{P}^n has nonzero homology of exactly \mathbb{Z} in all even real dimensions, so any cycle Z (topological, analytic, or algebraic) of complex dimension r has a well-defined degree $\deg Z$ associated to it, defined to be the image of said chain in the $2r$ th homology group \mathbb{Z} . The following theorem (the first step in the proof) gives a bound on the intersection multiplicity at a point of a piece of an analytic set of dimension r with an analytic cycle of dimension $n - r$:

Theorem 2.1. *Let E be an analytic simplex of dimension r that is an element in a triangulation of a projective closed irreducible analytic set W (also of dimension r), and let p be an interior point of E . Then there exists a positive integer N such that if Z is an analytic cycle of dimension $n - r$ such that p is an isolated intersection point of E with Z , then the multiplicity of the intersection is bounded by $N \cdot \deg Z$.*

Extremely sketchy proof, in bullet form.

- Reduce immediately to the case that Z is a closed irreducible analytic set itself.
- Let $\mathrm{PGL}_{n+1}(\mathbb{C})$ act on \mathbb{P}^n in the usual way. Prove that the set of $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$ such that $g \cdot W$ and Z have only regular intersection points is a dense open set. (This step makes precise the notion that regular intersections should be a generic property of sets that are nice enough. The proof is rather messy and uses the Baire category theorem on several occasions,

as one might expect.) Use this to find a g very close to the identity such that $g \cdot W \cap Z$ consists only of regular intersection points.

- Prove that regular intersection points for finite analytic complexes³ have intersection multiplicity one (in particular, not negative one, since we've chosen our orientation coherently).
- By the above two bullet points and general topology, the intersection of $g \cdot W$ and Z consists of precisely $\deg W \cdot \deg Z$ points.
- Prove that given a "small" perturbation of a chain, such as the perturbation by g , we can find a small neighborhood U of p , containing no other intersection points, such that the total intersection multiplicity inside U stays the same (loosely speaking, intersections inside U stay inside U , and intersections outside U stay outside U). Therefore

$$\#\{q \in g \cdot W \cap Z \cap U\} = \text{intersection multiplicity of } p.$$

- Since the right hand side above is certainly bounded by $\deg W \cdot \deg Z$, we have proven the theorem with $N = \deg W$. \square

Step 1 having been completed, we move onto Step 2.

Theorem 2.2. *Let E be an analytic simplex of dimension r in \mathbb{P}^n that is not an element of a triangulation of an algebraic variety. Then given any positive integer N , there exists a positive algebraic cycle Z of dimension $n - r$ such that there is an isolated intersection point of E and Z with multiplicity greater than $N \cdot \deg Z$.*

If we can prove this theorem, Chow's theorem follows immediately: if E is an analytic simplex that is an element in a triangulation of a closed irreducible analytic set W , then using Theorems 2.1 and 2.2, we find that E must be an element of a triangulation of a closed irreducible algebraic set V . By analytic continuation, they coincide. We can do this procedure for each irreducible branch of our analytic set to show that it is algebraic.

As for the proof of Theorem 2.2, we will make ourselves content with an analogy. Consider the following theorem: a number ζ is algebraic if and only if there exists a positive integer N such that the inequality $|\sum_{i=0}^n z_i \zeta^i| < |Z|^{-N}$ has only finitely many solutions, where $Z = \max\{|z_0|, |z_1|, \dots, |z_n|\}$. We can rephrase this as the statement that for every n , there is a constant Γ_n such that

$$\left| \sum_{i=0}^n z_i \zeta^i \right| < (\Gamma_n Z)^{-N}$$

has no solution with a nonzero left hand side. Taking the logarithm, we get

$$-\log \left| \sum_{i=0}^n z_i \zeta^i \right| > N(\log Z + \log \Gamma_n).$$

Now consider the following analogy, which is loosely the number field/function field analogy: ζ corresponds to an irreducible analytic curve, whereas a polynomial $\sum_{i=0}^n z_i y^i$ corresponds to an irreducible algebraic curve. Then one can view $\log Z +$

³This is the place where we have to use the closed hypothesis, which corresponds to our being able to take a finite triangulation of the analytic variety.

$\log \Gamma_n$ as some sort of “degree” of the algebraic curve, and

$$-\log \left| \sum_{i=0}^n z_i \zeta^i \right|$$

as the “intersection multiplicity” of the algebraic curve $\sum_{i=0}^n z_i y^i$ and the irreducible analytic curve ζ .

Under this correspondence, the above theorem in diophantine analysis says that an analytic curve E is algebraic if and only if there is a number N such that no algebraic curve of degree d can have an isolated intersection with E with a multiplicity greater than $N \cdot d$. Just as how one cannot approximate algebraic numbers too well by real numbers, one cannot approximate algebraic varieties too well by analytic varieties. This intuition makes the proof of Theorem 2.2, while still ugly, at least tolerable.

3. VIA REMMERT-STEIN

Four years after Chow, Remmert and Stein found an alternative path to Chow’s theorem, using a theorem that is rather important in its own right. To illustrate this method, I’ll state the Remmert-Stein theorem, explain a bit of how one would go about proving it, and then legitimately deducing Chow’s theorem from Remmert-Stein.

Before we begin, we have to (re)define the dimension of an analytic space⁴. There is a basic general theory that an analytic space X consists of regular points $\mathbf{R}(X)$ and singular points $\mathbf{S}(X)$, the former of which is open and the latter of which is nowhere dense. At each point $z \in \mathbf{R}(X)$, use the local model to assume that the set is an analytic subset of \mathbb{C}^n . Then z is a regular point if there exists a neighborhood of z in which the analytic set is given precisely as the vanishing locus of a finite r -tuple of functions whose Jacobi matrix at z has full rank r . The dimension of X at the regular point $\dim_z X$ is then defined to be r , and we define

$$\dim X = \sup_{z \in \mathbf{R}(X)} \dim_z X.$$

By a *polydisc* in \mathbb{C}^m , we just mean a product of open discs in the complex plane.

Theorem 3.1 (Remmert-Stein theorem). *Let V be an analytic subvariety of a polydisc $D \subseteq \mathbb{C}^m$, and let W be an irreducible analytic subvariety of $D \setminus V$. If $\dim W > \dim V$, then the closure \overline{W} in D is an irreducible analytic subvariety, and $\dim \overline{W} = \dim W$.*

This seems like a manifestly reasonable theorem; how bad could an analytic subvariety possibly be that its closure would no longer have the same dimension, or even be a variety? In fact, it’s not at all obvious, and the dimension inequality $\dim W > \dim V$ is absolutely essential. We can cook up a couple of enlightening counterexamples.

First, a simple example to show that the closure to an analytic variety in a larger space need not be an analytic variety. Let $D = \mathbb{C}^2$ with coordinates z_1, z_2 ,

⁴This is the same notion as used above, where dimension was defined in terms of the triangulation.

let $V = \{z_1 = 0\}$, and let

$$W = \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, 0 \right) \right\}.$$

This is an analytic variety in $D \setminus V$, since we can draw a neighborhood around each isolated point and define it locally and analytically. Its closure \overline{W} in D , however, is *not* an analytic variety: any neighborhood around the origin contains too many zeros to be cut out by a nonzero holomorphic function (i.e., the set of zeros has a limit point).

Okay, this is a bit of a silly example; in particular, W is not remotely irreducible here. Here's another example with an irreducible W that uses the weirdness of an essential singularity. Let $D = \mathbb{C}^2$ with coordinates z_1, z_2 , let $V = \{z_1 = 0\}$ and let $W = \{z_2 = e^{1/z_1}, z_1 \neq 0\}$, which is easily checked to be an analytic subvariety of $D \setminus V$. But using the dimension theory of analytic varieties, \overline{W} cannot possibly be an analytic subvariety of D . If it were, it would have to be one-dimensional, and its intersection with $\{z_2 = 1\}$, for example, would be zero-dimensional and thus consist only of isolated points. But by Picard's theorem, $(0, 1)$ is a limit point of the intersection, not an isolated point. We conclude that the inequality $\dim W > \dim V$ is not removable in our hypotheses.

Proof of Remmert-Stein (sketch). We will introduce two notions and assume a basic theorem about each.

The first notion is that of a certain set of holomorphic functions being complete. Let F be a collection of elements in \mathcal{H}_X^d (i.e., d -tuples of global sections) on an analytic space X . Define the level set of F at x to be

$$L_x(F) = \{y \in X : f(y) = f(x) \text{ for all } f \in F\}.$$

We say X is F -complete if $\dim_x L_x(F) = 0$ for all $x \in X$. That is, the simultaneous level sets of all functions in F are zero-dimensional, or equivalently consist of a discrete set of points. The notion of completeness applies equally well to a single function, rather than sets of functions. The main theorem about completeness is the following statement:

Theorem 3.2. *Let (X, \mathcal{H}_X) be an analytic space of pure dimension d , and \mathcal{F} a Fréchet algebra of global sections on X such that the injection $i : \mathcal{F} \rightarrow \mathcal{H}_X$ is continuous. Suppose X is \mathcal{F} -complete. Then there exists a $g \in \mathcal{F}^d$ such that X is g -complete. \square*

One can actually prove something much stronger than this: the set of such g is a set of second category.

The second notion is that of an *analytic cover*, which is a triple (X, π, U) satisfying the following conditions:

- (i) X is a locally compact Hausdorff space,
- (ii) U is a domain in \mathbb{C}^n ,
- (iii) π is a proper mapping of X onto U with discrete fibers,
- (iv) there is a negligible set $A \subset U$ and a natural number λ such that π is a λ -sheeted covering map $X \setminus \pi^{-1}(A) \rightarrow U \setminus A$, and
- (v) $X \setminus \pi^{-1}(A)$ is dense in X .

Here, a *negligible set* is a nowhere dense set such that all bounded holomorphic functions defined away from the set can be extended across it. We let $X_0 = X \setminus$

$\pi^{-1}(A)$, so in particular π is a local homeomorphism on X_0 . Given an analytic cover, we can define an analytic structure on X by pulling back along π ; that is, given an open set $W \subseteq X$ and a function f on W we call f holomorphic if for every $W' \subseteq X_0$ on which π is a homeomorphism we have that $f|_{W'} \circ \pi^{-1}$ is holomorphic on $\pi(W') \subseteq U$. The utility of analytic covers is that they provide a different way of thinking about analytic varieties (rather than as an “admissible representation”). It turns out that an irreducible variety is locally an analytic cover, and conversely that an analytic cover is itself an analytic variety (the latter is a hard theorem of Grauert and Remmert). For our purposes, we need the following theorem, which is not difficult:

Theorem 3.3. *Let D be a domain in \mathbb{C}^n , $X \subset D$ a subset, and $g : D \rightarrow \mathbb{C}^k$ a holomorphic mapping such that $(X, g, g(D))$ is an analytic cover and X_0 is a complex submanifold of \mathbb{C}^n . Then X itself is an analytic subvariety of D . \square*

The idea of the proof of Remmert-Stein is to set up all of these conditions with $X = \overline{W}$. It suffices to consider the situation locally: we want to show that for every $x \in \overline{W}$, there is a neighborhood Δ of x such that $\overline{W} \cap \Delta$ is a subvariety of Δ . If $x \notin V$, there is nothing to prove, so we can assume $x \in \overline{W} \cap V$. Let W' be any variety of the same dimension as W that contains V in some neighborhood U of x (such a variety is easily found). Abstractly, take the disjoint union restricted to U ,

$$W^0 = (W' \sqcup W)|_U,$$

and the ordinary union restricted to U ,

$$\tilde{W} = (W' \cup W)|_U.$$

If we let $\mathcal{F} = \mathcal{H}_U / \mathcal{A}_{\tilde{W}}$, then we can consider \mathcal{F} as a Fréchet algebra on W_0 with a topology finer than that of \mathcal{H}_{W_0} . Therefore W^0 is \mathcal{F} -complete, so by Theorem 3.2 there is an $F \in \mathcal{H}_U^n$ such that F has zero-dimensional level sets on W^0 . We can assume without loss of generality that $F(x) = 0$.

We can therefore easily restrict to a small enough disc so that $F \neq 0$ on the boundary (because F has zero-dimensional level sets) and F is proper. The remaining steps are to strip away that “bad parts” of F to get an analytic cover. At each step, we need to show that the part we are throwing away is a negligible set. The three steps are:

- (1) Cut out $F^{-1}(F(V))$ from the domain. This is negligible because $\dim V < \dim W$. Call the new domain W_1 .
- (2) Cut out $F^{-1}(F(\mathbf{S}(W_1)))$ from the domain, where we recall that $\mathbf{S}(W_1)$ are the singular points of W_1 . This is negligible because $F(\mathbf{S}(W_1))$ is a subvariety of dimension less than $\dim W$. Call the new domain W_2 .
- (3) Cut out the points of the domain where the rank of F is less than full. Call the new domain W_3 .

In all, we’ve thrown away a negligible set and we’re left with a proper, nonsingular map from the rest to some dense set of a polydisc. This map is therefore a covering map, so we can apply Theorem 3.3 to conclude that \overline{W} is a variety. \square

As a corollary, if X is an analytic space, Y an analytic subvariety of X , and W an analytic subvariety of $X \setminus Y$ with finitely many branches each with dimension greater than $\dim Y$, then \overline{W} is an analytic subvariety of X . This follows because we can consider each branch separately.

It remains to show that the Remmert-Stein theorem implies Chow's theorem.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the usual projection to homogeneous coordinates, and consider an analytic subvariety V of \mathbb{P}^n . Since π has rank n everywhere, $\pi^{-1}(V)$ has dimension ≥ 1 everywhere (in fact, π^{-1} is clearly a cone missing the origin). Since V itself is compact, being a closed subset of a compact space \mathbb{P}^n , it has finitely many branches, so $\pi^{-1}(V)$ has finitely many branches as well. By (the corollary to) Remmert-Stein, $V' = \pi^{-1}(V) \cup \{0\}$ is an analytic variety of \mathbb{C}^{n+1} . Geometrically, V' is a cone (including the origin).

Consider the germ V'_0 of V' at 0, which is the vanishing set of some g^1, \dots, g^k with $g^i \in \mathcal{H}_{n+1,0}$ (i.e., germs of holomorphic functions at the origin). Expand each g^i into homogeneous polynomials:

$$g^i = \sum_{n=1}^{\infty} g_n^i.$$

Then

$$g^i(tz) = \sum_{n=1}^{\infty} g_n^i(z)t^n$$

for all $z \in \mathbb{C}^{n+1}$, $t \in \mathbb{C}$. If $z \in V'$, then since V' is a cone $tz \in V'$ for all $t \in \mathbb{C}$, so $g^i(tz) = 0$ identically. By the above relation, all coefficients $g_n^i(z)$ vanish. So in fact V'_0 is the vanishing set of all the homogeneous parts g_n^i , which are polynomials. By Noetherianity, finitely many such suffice, and we're done.

4. VIA SERRE

Serre's proof of Chow's theorem comes as a corollary of a much further-reaching set of theorems, known collectively as the GAGA results. I will state the results and describe the methods by which they are proved⁵. Our focus changes from analytic to sheaf-theoretic, as the developments of the early 1950s (especially in Serre's own FAC paper) bear fruit.

The basic setup is as follows. Let (X, \mathcal{O}_X) be a closed algebraic subvariety of \mathbb{P}^n . We can also put an analytic structure on X , which will have a different underlying topological space (inherited from the Euclidean topology) and a different structure sheaf \mathcal{H}_X . For ease of notation we denote this analytic space by X^h . Every algebraic morphism $f : X \rightarrow Y$ is also an analytic morphism, and this analytification commutes with various operations, such as restriction to a closed subvariety and taking the product of two varieties. One can prove, though it is not immediately obvious, that analytification preserves dimension.

We now wish to talk about sheaves on these two structures; i.e., sheaves of \mathcal{O}_X -modules and of \mathcal{H}_X -modules. Given a sheaf \mathcal{F} on \mathcal{O}_X , we pull back along the obvious continuous map $c : X^h \rightarrow X$ and then tensor with \mathcal{H}_X to get the analytification:

$$\mathcal{F}^h = c^* \mathcal{F} \otimes_{c^* \mathcal{O}_X} \mathcal{H}_X$$

In the same way, every homomorphism of sheaves can get promoted to a homomorphism of the analytification of said sheaves, and it is immediate that analytification is a functor on sheaves.

⁵Though it's probably better for everyone to just read the GAGA paper, given how well-written it is.

It takes some nonzero quantity of algebra to check that analytification of sheaves is actually an exact functor; in fact, Serre introduced the concept of flatness, and checked that the completion of a Noetherian ring is flat over that ring, for this very purpose. In his language, one first verifies that the completions of \mathcal{O}_X and \mathcal{H}_X coincide⁶ to conclude that the corresponding local rings are a “flat couple” (“couple plat”) and thenceforth to check exactness. From exactness and Oka’s theorem, which states that the structure sheaf of an analytic variety is coherent⁷, we conclude that analytification preserves coherence. There is an induced functor $H^q(X, \mathcal{F}) \rightarrow H^q(X^h, \mathcal{F}^h)$ on cohomology groups in the usual way.

We can now state the main results of GAGA:

Theorem 4.1 (GAGA). *Let X be a closed algebraic variety in \mathbb{P}^n .*

- (1) *For each $q \geq 0$ and coherent algebraic sheaf \mathcal{F} on X , the induced functor $H^q(X, \mathcal{F}) \rightarrow H^q(X^h, \mathcal{F}^h)$ is an isomorphism.*
- (2) *Let \mathcal{F} and \mathcal{G} be two coherent algebraic sheaves on X . Then every analytic sheaf homomorphism $\mathcal{F}^h \rightarrow \mathcal{G}^h$ comes from a unique algebraic sheaf homomorphism $\mathcal{F} \rightarrow \mathcal{G}$.*
- (3) *Let \mathcal{M} be a coherent analytic sheaf on X^h . Then there exists a coherent algebraic sheaf \mathcal{F} on X such that $\mathcal{F}^h \simeq \mathcal{M}$, unique up to unique isomorphism.*

Proof sketch. For part (1), first extend the sheaves \mathcal{F} and \mathcal{F}^h by zero to all of \mathbb{P}^n . By a result in Serre’s FAC, this does not change the cohomology. In the case where $\mathcal{F} = \mathcal{O}$ (so $\mathcal{F}^h = \mathcal{H}$), we can show that the cohomology is equal in degree zero for stupid reasons and equal in all higher degrees because all groups vanish by comparison to Dolbeault cohomology and Dolbeault’s theorem. Next we attack the case $\mathcal{F} = \mathcal{O}(r)$. We induct on dimension by restricting to a hyperplane E , getting the short exact sequence

$$0 \rightarrow \mathcal{O}(r-1) \rightarrow \mathcal{O}(r) \rightarrow \mathcal{O}_E(r) \rightarrow 0.$$

Take the long exact sequence in both cohomologies, where the vertical arrows are the aforementioned induced functors:

$$\begin{array}{ccccccc} H^q(X, \mathcal{O}(r-1)) & \longrightarrow & H^q(X, \mathcal{O}(r)) & \longrightarrow & H^q(X, \mathcal{O}_E(r)) & \longrightarrow & H^{q+1}(X, \mathcal{O}(r-1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q(X^h, \mathcal{O}(r-1)^h) & \longrightarrow & H^q(X^h, \mathcal{O}(r)^h) & \longrightarrow & H^q(X^h, \mathcal{O}_E(r)^h) & \longrightarrow & H^{q+1}(X^h, \mathcal{O}(r-1)^h) \end{array}$$

Using the five lemma, we find that everything in sight is an isomorphism. Finally, for the general case write \mathcal{F} as a quotient of a direct sum \mathcal{L} of $\mathcal{O}(r)$ (we can do this by a result in FAC), getting some short exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{L} \rightarrow \mathcal{F} \rightarrow 0.$$

By again taking the long exact sequence, using the five lemma, and inducting downwards on the cohomology dimension, we get the desired result.

⁶This is at least heuristically plausible when one notes that the completion of a polynomial ring and the ring of convergent power series coincide; namely, they are both the ring of formal power series. However the general result is nontrivial, requiring the use of the “holomorphic nullstellensatz.”

⁷This is nontrivial. The Weierstrass Preparation Theorem is essential.

For part (2), set $\mathcal{A} = \mathcal{H}om(\mathcal{F}, \mathcal{G})$ and $\mathcal{B} = \mathcal{H}om(\mathcal{F}^h, \mathcal{G}^h)$. There is an obvious inclusion map from \mathcal{A} into \mathcal{B} , which we can extend by linearity to get a map $i : \mathcal{A}^h \rightarrow \mathcal{B}$. Examine the following diagram:

$$H^0(X, \mathcal{A}) \xrightarrow{\epsilon} H^0(X^h, \mathcal{A}^h) \xrightarrow{i} H^0(X^h, \mathcal{B}).$$

The goal is precisely to see that the composition of these two maps is an isomorphism. We know that ϵ is an isomorphism by part (1). We can show that i is an isomorphism by checking on stalks: by coherence, we can pull the tensor product outside Hom and then do some easy algebra (using again that $(\mathcal{O}_x, \mathcal{H}_x)$ is a “flat couple”). This yields the result.

Part (3) is more complicated and I will be even sketchier. Consider some analytic sheaf \mathcal{M} on X^h . Uniqueness of a possible algebraic \mathcal{F} such that $\mathcal{F}^h = \mathcal{M}$ follows, after some trickiness, from part (2). Because analytification plays well with restriction, it suffices to prove for $X = \mathbb{P}^n$.

We will need “baby versions” of Cartan’s Theorems A and B. If \mathcal{A} is a sheaf on X^h , define its twist to be

$$\mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{H}} c^* \mathcal{O}(n)$$

Then our version of Cartan A is that for any hyperplane E and coherent analytic sheaf \mathcal{A} on E , one has $H^q(E^h, \mathcal{A}(n)) = 0$ for $q > 0$ and all sufficiently large n . Cartan B is the statement that for all sufficiently large n , stalks of $\mathcal{M}(n)$ are generated by global sections; i.e., for all $x \in \mathbb{P}^n$, $\mathcal{M}(n)_x$ is generated by $H^0(\mathbb{P}^n, \mathcal{M}(n))$ as an \mathcal{H}_x -module.⁸ I will assume these theorems; they can be proven using slick cohomological and linear-algebraic arguments (and are so proven in Serre’s paper). The Theorem B requires the Cartan-Serre theorem that the dimension of cohomology of coherent sheaves on projective spaces is finite.⁹

Given these facts, we can finish the proof. By Theorem B, $\mathcal{M}(n)$ is a quotient of \mathcal{H}^p for some p , so \mathcal{M} is a quotient of $\mathcal{H}(-n)^p$. Let $\mathcal{L}_0 = \mathcal{O}(-n)^p$, so we have an exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{L}_0^h \rightarrow \mathcal{M} \rightarrow 0.$$

Applying the same logic to \mathcal{R} , we get

$$\mathcal{L}_1^h \xrightarrow{g} \mathcal{L}_0^h \rightarrow \mathcal{M} \rightarrow 0$$

for some algebraic sheaf \mathcal{L}_1 . Using (2), find an $f : \mathcal{L}_1 \rightarrow \mathcal{L}_0$ such that $g = f^h$. Let \mathcal{F} be the cokernel of f , so we get

$$\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0.$$

Analytifying the above sequence, we find that $\mathcal{F}^h \simeq \mathcal{M}$, as desired. \square

Now we can prove Chow as a corollary. Let X be a closed analytic subset of \mathbb{P}^n , and let \mathcal{M} be $\mathcal{H}_{\mathbb{P}^n}$ modulo the sheaf of analytic ideals on X^h . By part (3) of GAGA, there exists a coherent algebraic sheaf \mathcal{F} on X such that $\mathcal{F}^h = \mathcal{M}$. It is easy to see that the support of \mathcal{F}^h is equal to the support of \mathcal{F} (corresponding to

⁸In general, Cartan’s Theorems A and B are much more general statements that higher cohomology vanishes and stalks are generated by global sections for all Stein spaces, which is a fairly general class of analytic spaces. These “souped-up” theorems require much more analytic work to prove.

⁹Interestingly, in the more general setting of Stein spaces, it is cleanest to prove this result as a (nontrivial) corollary of Theorems A and B.

the algebraic fact that $c^*\mathcal{F} \rightarrow \mathcal{F}^*$ is an injection). This support, X , is therefore Zariski-closed (because \mathcal{F} is coherent) so we're done.

5. SOME CONSEQUENCES

- By applying Chow's theorem to the graph of a holomorphic map from a compact projective variety into an algebraic variety, we find rather quickly that every such map is actually an algebraic morphism.
- Every meromorphic function on \mathbb{P}^n is rational (generalizing the fact for \mathcal{P}^1). The proof of this that I have seen pulls back to the affine cone of \mathbb{P}^n and uses Hartogs' theorem.
- Chow's theorem is one possible way to prove that every compact Riemann surface is an algebraic curve, by first showing that there is an analytic embedding into \mathbb{P}^n . The "standard" way to do this is via the Riemann-Roch theorem, but there are also analytic arguments (i.e., Hörmander's L^2 methods).

6. REFERENCES AND FURTHER READING

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