13 Lecture 13: Uniformity and sheaf properties

13.1 Introduction/global theory

So far, we have avoided discussing the global theory of adic spaces, but this is a convenient point to introduce the main ideas. Recall that for any Huber pair (A, A^+) we have defined a presheaf of complete topological rings $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$: we first defined

$$\mathcal{O}_X(X(T/s)) = A\langle T/s\rangle$$

for a finite subset $T \subset A$ and $s \in S$ such that $T \cdot A$ is open in A, and then we define for an arbitrary open set U

$$\mathcal{O}_X(U) = \varprojlim_{W \subset U} \mathcal{O}(W),$$

where W varies through rational domains contained in U. (We also defined a presheaf $\mathcal{O}_X^+$ in the obvious parallel way.) Proposition 12.3.3 verified that this was well-defined; i.e., that for a given rational domain W its associated Huber pair is independent of the presentation $W = X(T/s)$. Proposition 12.3.7 verified that if we define the stalk $\mathcal{O}_{X,x}$ at a point $x \in X$ as the usual direct limit in the category of rings (we throw away topological information), it is a local ring on which there is a canonical valuation v_x.

The particular way that we have defined $\mathcal{O}_X(U)$ for general open U and the fact that the rational domains (by definition) form a basis for the topology on X together imply that $\mathcal{O}_X$ is *adapted* to the basis of rational domains, in Wedhorn's terminology [Wed, §8.9]. In fact, the categories of sheaves on a topological space and sheaves on a base of the topological space are equivalent.[1] In particular, ours is the unique way to extend the definition of a Huber pair associated to a rational domain to all open sets compatibly with restriction maps. Thus for any presheaf $\mathcal{F}$ that is adapted to a basis $\mathscr{B}$ on a topological space, $\mathcal{F}$ is a sheaf if and only if it is a sheaf when restricted to $\mathscr{B}$; for the purposes of checking the sheaf axioms, we can forget about any open sets that are not rational domains. This is excellent news; otherwise, any explicit calculation would be hopeless.

In order to define global adic spaces, the general strategy is obvious: we want to say that something is an adic space if it is locally of the form $\mathrm{Spa}(A, A^+)$, where (A, A^+) is a Huber pair. There are two problems in making this work: first, we need to figure out in what underlying category we are working (for example, in the case of schemes this is the category of locally ringed topological spaces, in the case of formal schemes this is the category of locally topologially ringed topological spaces, and in the case of rigid spaces this is the category of locally ringed "G-topologized" spaces). Second, perhaps with these examples in mind, we note that in order to get any sort of reasonable theory off the ground we had better ensure that our structure presheaves are, in fact, sheaves. Otherwise, we will be stuck in applying any sort of gluing or cohomological argument; one could argue that the resulting object would not be properly geometric at all. We will deal with the second problem by mandate, simply saying that an adic space has a structure sheaf by definition and thereby punting the hard work to the verification that the structure presheaf on $\mathrm{Spa}(A, A^+)$ actually forms a sheaf for certain useful classes of Huber pairs (A, A^+).

Here's a solution to the first problem:

[1] See [EGA, III_0 3.2.2]. See also [SP, Tag 009R], though the result is only stated there for so-called "sheaves of algebraic structures," which does not cover the case of topological rings. Stacks project exercise: generalize as much as possible of the theory in the chapter "Sheaves on Spaces" to a setting which includes sheaves of topological rings and add it to the Stacks Project.

Definition 13.1.1 Let V be the category of triples $(X, \mathcal{O}_X, (v_x)_{x\in X})$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings such that each stalk $\mathcal{O}_{X,x}$ (as defined in the category of rings) is a local ring, and v_x is an equivalence class of valuations on $\mathcal{O}_{X,x}$ such that $\text{supp}(v_x)$ is equal to the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$.[2] Morphisms $f : X \to Y$ in V are pairs $(f, f^\flat)$ where f is a continuous map of topological spaces, and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of topological rings such that the induced homomorphism on stalks is compatible with the valuations (in particular, the induced homomorphism on stalks is local).[3]

Definition 13.1.2 Given an element $(X, \mathcal{O}_X, (v_x)_{x\in X}) \in V$, we define the presheaf of integral structures $\mathcal{O}_X^+$ as follows. For any open U, set

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : v_x(f) \leq 1 \text{ for all } x \in U\}.$$

Remark 13.1.3 By Lemma 12.3.9, this definition agrees with the definition of the presheaf $\mathcal{O}_A^+$ on $\text{Spa}(A, A^+)$.

Definition 13.1.4 An *affinoid adic space* is an element of V isomorphic to $\text{Spa}(A, A^+)$ (with its associated structure sheaf and local valuations). An *adic space* is an element X of V such that there exists an open covering $\{U_i\}$ of X such that $X|_{U_i}$ is an affinoid adic space for every i.

In particular, by the definition of V, $\mathcal{O}_A$ is required to be a sheaf in order for $\text{Spa}(A, A^+)$ to be an adic space. If this is the case, we call the Huber pair (A, A^+) *sheafy.*

With these definitions in place, we could begin doing geometry in earnest, *if* we had a good way in any particular case to verify that (A, A^+) is sheafy. Thus our goal is to show that certain nice enough Huber pairs are sheafy so that the above definitions are not empty.

13.2 A note on the category V

We see that we need to include the data of the local valuations v_x in our definition of the category V in order to keep track of the integral structure. Without this information, we would not have the following proposition, which is the analogue of the fact that the opposite of the category of rings embeds fully faithfully into the category of schemes. Note that we must enforce the hypothesis of sheafiness for obvious reasons, and additionally due to the insensitivity of $\text{Spa}(A, A^+)$ to completions we add a completeness hypothesis.

Proposition 13.2.1 (Proposition 2.1 in [H2], Proposition 8.18 in [Wed]) *Let (A, A^+) and (B, B^+) be sheafy Huber pairs with B complete. Then the obvious map*

$$\text{Hom}((A, A^+), (B, B^+)) \to \text{Hom}(\text{Spa}(B, B^+), \text{Spa}(A, A^+))$$

is a bijection. Here morphisms of Huber pairs are continuous homomorphisms $f : A \to B$ such that $f(A^+) \subseteq f(B^+)$, and morphisms on the right hand side are morphisms of adic spaces (i.e., morphisms in the category V).

In other words, the embedding from the opposite of the category of complete sheafy Huber rings into the category of adic spaces is fully faithful.

Example 13.2.2 To see the importance of preserving the integral structure in the definition of a morphism of Huber pairs, consider the following example. Let k be an algebraically complete

[2] Equivalent to the last condition is the condition that v_x is continuous with respect to the $\mathfrak{m}$-adic topology on $\mathcal{O}_{X,x}$.

[3] Equivalently, $f^\flat$ is required to satisfy $f^\flat(\mathcal{O}_Y^+) \subseteq f^\flat(\mathcal{O}_X^+)$ and be a local morphism when restricted to these integral structures.

nonarchimedean field, let $\mathbf{D}_k = \mathrm{Spa}(k[t], k^0[t])$ be the usual adic unit disc, and let $\mathbf{D}'_k = \mathrm{Spa}(k[t], A^+)$ where

$$A^+ = k^0 + t\mathfrak{m}[t].$$

This example was discussed in 11.3.13; the conclusion was that $\mathbf{D}'_k$ consists of the union of $\mathbf{D}_k$ with one additional type 5 point $v_{0,1^+}$ whose generization is the Gauss point. The identity map $k[t] \to k[t]$ induces a morphism of Huber pairs $(k[t], A^+) \to (k[t], k^0[t])$, which corresponds to the inclusion $\mathbf{D}_k \hookrightarrow \mathbf{D}'_k$. However, the identity does not induce a morphism of Huber pairs $(k[t], k^0[t]) \to (k[t], A^+)$, which is good, as clearly $v_{0,1^+}$ has nowhere to go in the purported corresponding map $\mathbf{D}'_k \to \mathbf{D}_k$.

13.3 Huber's criteria

The first sheafiness criteria were proven by Huber. In addition to proving the sheaf property, he also proves that higher cohomology of the structure sheaf vanishes on every rational subdomain, which is essential if one wants to calculate anything with sheaf cohomology.

Theorem 13.3.1 (Theorem 2.2 in [H2]) *Let (A, A^+) be a Huber pair. Then $\mathcal{O}_A$ is a sheaf, and its higher cohomology vanishes on rational subdomains, if either of the following conditions holds:*

(i) $\hat{A}$ has a noetherian ring of definition.

(ii) A is a strongly noetherian *Tate ring, meaning that all relative Tate algebras over A are noetherian (equivalently, all rings $A\langle t_1, \ldots, t_n\rangle$ are noetherian).*

Recall that a Huber ring is Tate if it contains a topologically nilpotent unit. It is apparently not known whether there exist complete Tate rings that are noetherian but not strongly noetherian. An example of a (noncomplete) Tate ring that is noetherian but not strongly noetherian is given by the non-sheafy ring A in Example 13.5.1 below (if it were strongly noetherian, it would be sheafy by 13.3.1!).

To appreciate the significance of these two conditions, recall that one of Huber's motivations in the definition of the category of adic spaces was to construct a category that fully faithfully contained the categories of locally noetherian formal schemes and Tate's rigid analytic spaces. We have not defined the functors that realize these embeddings, but when we do we will see that (i) applies to all adic spaces that come from locally noetherian formal schemes, and (ii) applies to all adic spaces that come from rigid analytic spaces. In particular, given the obvious functor from locally noetherian schemes to locally noetherian formal schemes that assigns the discrete topology to each quasi-compact open set,[4] we see that the category of adic spaces fully faithfully contains the category of locally noetherian schemes.

Example 13.3.2 The adic space $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$ corresponding to the scheme $\mathrm{Spec}(\mathbf{Z})$ (so $\mathbf{Z}$ is given the discrete topology) turns out to be the final object in the category of adic spaces. There is exactly one valuation in $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$ above the generic point of $\mathrm{Spec}(\mathbf{Z})$ (the trivial valuation on $\mathbf{Q}$) and two valuations above each rational prime p of $\mathrm{Spec}(\mathbf{Z})$ (one corresponding to the trivial valuation on $\mathbf{F}_p$ and one corresponding to the p-adic valuation on $\mathbf{Q}_p$).

It should not be a surprise that (i) of Theorem 13.3.1 is proven using techniques in the cohomology of formal schemes borrowed from [EGA], while (ii) very closely follows Tate's original acyclicity result for rigid analytic spaces.

[4] and the so-called "pseudo-discrete" topology to each open set in general; for a discussion of this functor see [SP, Tag 0AHY].

The best example to keep in mind when considering the usefulness of (ii) is the adic space associated to the Tate ring $\mathbf{C}_p\langle t_1, \ldots, t_n\rangle$. Although this ring is noetherian (and very nice in several other respects as well), its topology is *not* generated by a noetherian ring of definition because $\mathbf{C}_p$ is not discretely valued. Tate rings over discretely valued fields such as $\mathbf{Q}_p$ are sheafy by both (i) and (ii).

Example 13.3.3 Although we will ultimately care more about criterion (ii) than (i), here is a useful toy example of the kinds of mixed-characteristic adic spaces that are now being considered in the theory of perfectoid spaces. This example was taken from [S2]. Let

$$X = \mathrm{Spa}(\mathbf{Z}_p[[T]], \mathbf{Z}_p[[T]]),$$

where $\mathbf{Z}_p[[T]]$ is given the (p,T)-adic topology. Everything in sight is noetherian, so (i) of Theorem 13.3.1 proves that X is an adic space. There exists exactly one non-analytic point s; if we consider the map of topological spaces $X \to \mathrm{Spec}(\mathbf{Z}_p[[T]])$ sending a valuation to its support, it is the unique closed point lying above $(p,T) \in \mathrm{Spec}(\mathbf{Z}_p[[T]])$. Explicitly, s is the valuation

$$\mathbf{Z}_p[[T]] \twoheadrightarrow \mathbf{F}_p \to \{0,1\},$$

where the first arrow sends both p and T to zero, and the second arrow is the trivial valuation on $\mathbf{F}_p$.

Now remove this point, defining

$$Y = X \setminus \{s\}.$$

Now Y consists only of analytic points. It possesses exactly one characteristic p point, corresponding to a copy of $\mathrm{Spa}(\mathbf{F}_p((T)), \mathbf{F}_p[[T]])$; all other points are characteristic zero. We can draw a useful picture of Y by considering the inclusions $\mathrm{Spa}(\mathbf{F}_p[[T]], \mathbf{F}_p[[T]]) \hookrightarrow X$ and $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p) \hookrightarrow X$, and imagining these two spaces as the horizontal and vertical axes of a plane, respectively. The horizontal axis is the locus "$T = 0$" while the vertical axis is the locus "$p = 0$" (i.e., the characteristic p part, or the special fiber). Both of these two spaces consist of exactly two points, and they intersect at the non-analytic point s, which we draw at the origin. All other points lie "somewhere between" the horizontal and vertical axes in the first quadrant of the plane. One can check that certain rational domains for Y, consisting of "rational sectors" of this quadrant, cover various subsets of interest: if we set

$$Y_n^+ = \{v \in Y : v(T^n) \leq v(p) \neq 0\} \quad \text{and} \quad Y_n^- = \{v \in Y : v(p^n) \leq v(T) \neq 0\},$$

where n varies over nonnegative integers, then $\bigcup_{n=1}^{\infty} Y_n^+$ is exactly the complement of the locus $p = 0$ and $\bigcup_{n=1}^{\infty} Y_n^-$ is exactly the complement of the locus $T = 0$. One can easily calculate that the complete Huber pairs associated to Y_n^+ and Y_n^- are $(A^+[1/p], A^+)$ and $(B^+[1/T], B^+)$, respectively, where A^+ is the p-adic completion of $\mathbf{Z}_p[[x]][x^n/p]$ and B^+ is the p-adic completion of $\mathbf{Z}_p[[x]][p^n/x]$. All of these Huber rings are Tate, though they do not contain any field.

The moral of these calculations is that the complement of the locus $p = 0$ in Y, which we would like to define as the generic fiber of Y,[5] is covered by a *countable union* of very nice rational domains, but is not itself a rational domain. In fact, it is not even quasi-compact, so it cannot possibly be an affinoid adic space. This is very different from the corresponding picture for schemes, where the generic fiber of an affine scheme over a mixed-characteristic ring is certainly affine! In particular, it would be a bad mistake to attempt to define the generic fiber as Spa of $\mathbf{Z}_p[[x]][1/p]$ with some topology and some ring of integers; as we have in fact already seen in Example 5.5.5.

[5] To make this picture clearer, one can consider Y as fibered over the adic space $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$ (our vertical axis), which consists of two points, one in characteristic p and one in characteristic zero. The fiber over the characteristic p point consists of the one point described above, while the fiber over the characteristic zero point is its complement, which we would like to define as the generic fiber of Y. One could define similar notions for the fibers of Y over our horizontal axis $\mathrm{Spa}(\mathbf{F}_p[[x]], \mathbf{F}_p[[x]])$.

As a final and somewhat parenthetical remark, the space X is a nice illustration of the following lemma, which states that a point is analytic if and only if it is "locally Tate":

Lemma 13.3.4 *Let (A, A^+) be a Huber pair and $x \in \mathrm{Spa}(A, A^+)$. Then x is analytic if and only if there exists a rational domain U containing x such that $\mathcal{O}_A(U)$ is Tate.*

Proof. First assume that x is analytic, and let $\{f_1, \ldots, f_n\}$ generate an ideal of definition I of a ring of definition of A. Without loss of generality, assume that the x-valuation $v_x(f_1)$ of f_1 is larger than the x-valuations of any of the other f_i. Because x is analytic, the support of v_x is not open, so $v_x(I)$ is nonzero. In particular, $v_x(f_1) \neq 0$. Consider

$$U = X\left(\frac{\{f_1, \ldots, f_n\}}{f_1}\right).$$

By construction, $x \in U$, and since we can write $f_i = \frac{f_i}{f_1} \cdot f_1$ for each i entirely inside the ring of integers, we know that the topology on

$$\mathcal{O}_A^+(U) = A\left\langle \frac{\{f_1, \ldots, f_n\}}{f_1} \right\rangle^+,$$

which is *a priori* I-adic, is in fact f_1-adic. Therefore f_1 is a topologically nilpotent unit on U, so $\mathcal{O}_A(U)$ is Tate.

In the other direction, we need to show that if x is analytic "from the point of view of U" it is also analytic "from the point of view of $\mathrm{Spa}(A, A^+)$." This is easy enough by tracking around ideals of definition: any ideal of definition J of a ring of definition of $\mathcal{O}_A(U)$ cannot be killed by v_x, but any ideal of definition I of a ring of definition of A generates J, so I cannot be killed by v_x either. Therefore x is analytic. □

13.4 The criterion of Buzzard and Verberkmoes

Another sheafiness criterion has been established for a particular class of Tate rings; namely, those that [BV] calls *stably uniform.* Most of our favorite examples of Tate rings contain a field; for example, affinoid algebras in rigid geometry or (as we will see) perfectoid algebras over perfectoid fields. Therefore [BV] work under that hypothesis. However, it is unnecessary for the proofs to assume the presence of a field, as we will indicate, and there are examples where it is helpful to work in greater generality:

Example 13.4.1 Perhaps the minimal example of a Tate ring that does not contain a field is $\mathbf{Z}[X]_{(X)}$ with the X-adic topology. A considerably less silly example is the mixed-characteristic Example 13.3.3. An example "from nature" is the adic space avatar of the Fargues-Fontaine curve.

For the rest of this section, let (A, A^+) be a Huber pair with A Tate.

Definition 13.4.2 The pair (A, A^+) is *uniform* if the set of power-bounded elements A^0 is bounded.

Example 13.4.3 Non-examples of uniform Tate rings include non-reduced rings like $\mathbf{Q}_p[\epsilon]/(\epsilon^2)$ with the p-adic topology. For such a ring we have

$$A^0 = \mathbf{Z}_p \oplus \mathbf{Q}_p \cdot \epsilon,$$

which is clearly not bounded as it contains an entire $\mathbf{Q}_p$-line.

Definition 13.4.4 The pair (A, A^+) is *stably uniform* if for all rational $U \subseteq \mathrm{Spa}(A, A^+)$, the ring $\mathcal{O}_A(U)$ is uniform.

Recall that the rings $\mathcal{O}_A(U)$ are automatically complete, but this is irrelevant as uniformity is preserved under completion. Clearly, being uniform does not depend on the choice of ring of integers A^+, although being stably uniform may. We will see an example of a stably uniform pair that is not uniform in the next section.

Theorem 13.4.5 (Theorem 7 in [BV]) *Let (A, A^+) be a stably uniform Huber pair with A Tate. Then $\mathcal{O}_A$ is a sheaf (and in particular,* $\mathrm{Spa}(A, A^+)$ *is an adic space), and its higher sheaf cohomology on rational domains vanishes.*

The proof will be sketched later, after we work with some concrete examples. The above statement differs from that in [BV] byis working in the generality of all stably uniform Tate rings and including the statement about vanishing cohomology, which follows using the method of proof of Theorem 2.4.23 in [KL] (which itself essentially follows Tate's original proof of the acyclicity theorem in classical rigid geometry).

The key usefulness of this result is that there are no finiteness assumptions whatsoever (aside from those in the definition of a Huber pair), so it can be applied to really huge rings like those that arise in the theory of perfectoid spaces. Useful rings that cannot be handled by the above theorem include most rings coming from formal schemes, which will not in general be Tate, and nonreduced rings coming from classical rigid geometry such as Example 13.4.3. Both of these types of rings are sheafy by Huber's result.

13.5 Examples and counterexamples

A natural reaction to Theorem 13.4.5 is to ask whether the stably uniform hypothesis on Tate rings is necessary; that is, can we come up with an example of a Tate ring that is not sheafy? The construction is not obvious, but one example is the following:

Example 13.5.1 ([BV, §§4.1-2]) Let

$$A = \mathbf{Q}_p[T, 1/T, Z]/(Z^2).$$

This is a pretty tame ring, finitely generated over a discretely-valued nonarchimedean field, but we are going to define a rather nasty topology on it. Let A_0 be the $\mathbf{Z}_p$-submodule of A generated by all elements of the form (i) $p^{|m|}T^m$ and (ii) $p^{-|m|}T^m Z$, for all $m \in \mathbf{Z}$. One checks easily that A_0 is actually a ring and that $A_0[1/p] = A$. Put the topology on A that has as a local basis at zero the subgroups $p^n A_0$ for $n \in \mathbf{Z}$, so that in particular A_0 is an open subring of A with ideal of definition (p) and we see that (A, A_0) is a Huber pair. Let

$$X = \mathrm{Spa}(A, A_0).$$

The space X has a cover by the rational domains

$$U = \{v \in X : v(T) \leq 1\} \quad \text{and} \quad V = \{v \in X : v(T) \geq 1\}.$$

Claim: the element $Z \in A$, considered as a global section on X, is nonzero, but its restrictions to both U and V are zero. If this is the case, $\mathcal{O}_A$ is clearly not a sheaf, as it fails the "uniqueness of gluing" (or "locality") axiom.

The proof of the claim is straightforward; we know that

$$\begin{aligned}\mathcal{O}_A(X) &= \varprojlim_n A/p^n A_0,\\ \mathcal{O}_A(U) &= \varprojlim_n A/(p^n A_0[T]), \text{ and}\\ \mathcal{O}_A(V) &= \varprojlim_n A/(p^n A_0[1/T]).\end{aligned}$$

We verify that $Z \neq 0$ in $\mathcal{O}_A(X)$ because it is not contained in $p^n A_0$ for any nonnegative n. But we claim that $Z \in p^n A_0[T]$ and $Z \in p^n A_0[1/T]$ for all nonnegative n, which implies that $Z = 0$ in both $\mathcal{O}_A(U)$ and $\mathcal{O}_A(V)$! Specifically, the added flexibility of working with T and $1/T$ is enough to write Z as p^n, multiplied by an element in A_0, multiplied by a monomial in T or a monomial in $1/T$:

$$Z = p^n \cdot (p^{-n}T^{-n}Z) \cdot T^n \quad \text{and} \quad Z = p^n \cdot (p^{-n}T^{n}Z) \cdot T^{-n}.$$

Note that $p^{-n}T^{-n}Z$ is a monomial of the form (ii) by taking $m = -n$, and $p^{-n}T^nZ$ is a monomial of the form (ii) by taking $m = n$.

In this example, there is nothing special about the field $\mathbf{Q}_p$; the same calculation works for any nonarchimedean field with topology given by a rank-one valuation. A similar construction also works over $\mathbf{Z}$; see Rost's example at the end of Section 1 of [H2].

If one actually calculates the rings of sections $\mathcal{O}_A(U)$ and $\mathcal{O}_A(V)$, one finds that

$$\mathcal{O}_A(U) = \mathbf{Q}_p\widehat{[T, 1/T]}$$

with a topology generated by the subring $\mathbf{Z}_p[T, p/T]$, and

$$\mathcal{O}_A(V) = \mathbf{Q}_p\widehat{[T, 1/T]}$$

with a topology generated by the subring $\mathbf{Z}_p[pT, 1/T]$. That is, U is the adic space associated to the annulus $\{|p| \leq |T| \leq 1\}$ and V is the adic space associated to the annulus $\{1 \leq |T| \leq |1/p|\}$. They do not glue properly, however; in particular, $\mathcal{O}_A(X)$ contains the nilpotent element Z.

One can "perfectify" this example to get a topological space that is locally perfectoid (in fact, having a cover by rational domains that are just perfectoid annuli) but not globally perfectoid (because perfectoid spaces are defined to be adic spaces, and are in particular sheafy): we let

$$A' = \left(\varinjlim_{T \mapsto T^p} \mathbf{Q}_p[T, 1/T] \right) [Z]/(Z^2),$$

(which is often abbreviated as $\mathbf{Q}_p[T^{1/p^\infty}, T^{-1/p^\infty}, Z]/(Z^2)$) and let A'_0 be the $\mathbf{Z}_p$-submodule of A' generated by elements of the form $p^{|n|}T^n$ and $p^{-|n|}T^nZ$, for all $n \in \mathbf{Z}[1/p]$. The example then goes through as before.

We have seen a failure of locality in the presheaf $\mathcal{O}_A$. It is also possible to exhibit a Tate ring A such that $\mathcal{O}_A$ fails the gluing axiom:

Example 13.5.2 ([BV, §4.4]) Let

$$A = \mathbf{Q}_p[T, 1/T, Z_1, Z_2, \ldots],$$

and take A_0 to be the $\mathbf{Z}_p$-submodule of A generated by all elements of the form $p^dT^a \prod_i Z_i^{e_i}$, for $a, d \in \mathbf{Z}$ and each $e_i \in \mathbf{Z}_{\geq 0}$, subject to the following conditions:

(i) If all $e_i = 0$, then $d = |a|$.

(ii) If $\sum_i e_i = 1$, then $d = |a| - 2\min\{\sum_i ie_i, |a|\}$.

(iii) Otherwise, $d = |a| - 2\sum_i ie_i$.

It is a considerably more tedious process this time to check that A_0 is in fact a ring, and $A_0[1/p] = A$. Once this has been done, we can as above define a topology on A generated by the basis at zero consisting of the subgroups $p^n A_0$ for $n \in \mathbf{Z}$. We let the rational subdomains U and V of $\operatorname{Spa}(A, A_0)$ be defined exactly as in the previous example.

The key point here is that the element $\sum_i Z_i$ converges on U and on V, but does not converge on all of X, so there are two local sections that agree on intersections but that do not glue. To verify the former claim, we note that for all $n \geq 0$,

$$Z_n = p^n \cdot (p^{-n}T^{-n}Z_n) \cdot T^n \in p^n A_0[T],$$

so $Z_n \to 0$ in $A_0[T]$ as $n \to \infty$ and therefore the (nonarchimedean) sum converges on U. Note that the element $p^{-n}T^{-n}Z_n$ satisfies condition (ii) above. Similarly,

$$Z_n = p^n \cdot (p^{-n}T^{n}Z_n) \cdot T^{-n} \in p^n A_0[1/T],$$

so $Z_n \to 0$ in $A_0[1/T]$ as $n \to \infty$ and therefore the sum converges on V. The element $p^{-n}T^nZ_n$ likewise satisfies condition (ii).

To verify that $\sum_i Z_i$ does not converge in $\mathcal{O}_A(X)$ is more complicated, but the idea is straightforward. We define functions $\rho_i : A \to \mathbf{Q}_p$ that send each element of A to its coefficient of Z_i. It is easy to check that each ρ_i is continuous, so it extends to a function on the completion $\mathcal{O}_A(X)$. Now I claim that for any $r \in \mathcal{O}_A(X)$, we must have

$$\lim_{i\to\infty} \rho_i(r) = 0.$$

This follows because r is the limit of a Cauchy sequence of elements of A, and each element of A has no Z_N coefficient at all for large enough N, so a simple approximation argument suffices. The hypothetical global section $\sum_i Z_i$ does not satisfy this condition, and therefore cannot exist.

It seems unlikely that there exists a *noetherian* counterexample to the gluing axiom.

Finally, one would like a counterexample proving that "stably uniform" cannot be replaced by "uniform" in the statement of Theorem 13.4.5, or failing that at least an example of a Tate ring that is uniform but not stably uniform, to show that the condition is not empty. The latter construction is easier and is described below; for an example of the former see [BV, §4.6].

Example 13.5.3 ([BV, §4.5]) To construct a uniform Tate ring that is not stably uniform, first let

$$\tilde{A} = \mathbf{Q}_p[T, 1/T, Z],$$

and let A_0 be the $\mathbf{Z}_p$-submodule generated by elements of A of the form $(pT)^a(pZ)^b$, where $b \geq 0$ and $a \geq -b^2$. Again, one verifies that A_0 is actually a ring, and one sets $A = A_0[1/p]$ (this time, $A \neq \tilde{A}$; we'll never get $1/T$ by inverting p, for instance). Topologize A using the subgroups p^nA_0 as before.

First claim: A is uniform. In fact, $A_0 = A^0$. According to Lemma 11 of [BV], which states that any compatible grading on A and A_0 induces a grading on A^0, to show this equality it suffices to check on each graded part separately. This is easy: if $\lambda \cdot (pT)^a(pZ)^b \in A^0$ with $\lambda \in \mathbf{Q}_p$, then the powers $\lambda^n(pT)^{an}(pZ)^{bn}$ are bounded. But such powers escape $p^{-N}A_0$ for every N unless $\lambda \in \mathbf{Z}_p$, which in turn implies that $\lambda \cdot (pT)^a(pZ)^b$ was in A_0 after all. We have proven that A is uniform.

Second claim: with U defined as usual to be $\{v \in \mathrm{Spa}(A, A_0) : v(T) \leq 1\}$, the ring $\mathcal{O}_A(U)$ is not uniform. This is true because now the element Z/p^n is power-bounded for all n, so in particular Z generates a whole $\mathbf{Q}_p$-line in $(\mathcal{O}_A(U))^0$. To verify, we calculate

$$\left(\frac{Z}{p^n}\right)^{n+1} = (pT)^{-n^2-2n-1}(pZ)^{n+1}T^{n^2+2n+1} \in \widehat{A_0[T]} = \mathcal{O}_A(U)^+.$$

As the $(n+1)$st power has landed us inside the "closed unit ball" defining the topology on $\mathcal{O}_A(U)$, we are certainly never going to get out again; that is, Z/p^n is power-bounded.

13.6 Sketch of the proof

There are two steps in the proof of Theorem 13.4.5, following Tate. First, one verifies the sheaf property for a very specific type of cover. Specifically we wish to verify that for a uniform and Tate Huber pair (A, A^+), the sheaf property holds with respect to covers of the form

$$U = \{v \in X : v(t) \geq 1\} \quad \text{and} \quad V = \{v \in X : v(t) \leq 1\},$$

where t is any element of A and $X = \mathrm{Spa}(A, A^+)$. Second, one verifies that we can reduce the general case to this specific case, this time under a stably uniform hypothesis.

The heart of the proof is the first step, so we will consider it in more detail. We will deal first with the corresponding sequence of uncompleted exact rings, which for the cover $\{U, V\}$ reads

$$0 \to A \xrightarrow{\epsilon} A\left(\frac{t}{1}\right) \oplus A\left(\frac{1}{t}\right) \xrightarrow{\delta} A\left(\frac{t}{1}, \frac{1}{t}\right) \to 0. \tag{1}$$

Here ϵ is the localization map to both factors and δ is simply subtraction of the second factor from the first; thus in particular all maps in the above sequence are continuous. The notation is as defined in Lectures 6 and 7; we have that, for example,

$$A\left(\frac{t}{1}\right) = A\left(\frac{\{1,t\}}{1}\right) = A[X]_{\{1,t\}}/(1 - X),$$

where $A[X]_{\{1,t\}}$ is the polynomial ring $A[X]$ equipped with the topology where a basis of open sets around zero is given by the sets

$$U[\{1,t\} \cdot X] := \left\{ \sum_{\text{finite}} a_i X^i : a_i \in t^i \cdot U \right\}$$

where U runs over all open additive subgroups $U \subseteq A$. This construction makes sense by Proposition 6.3.5 because $\{1, t\}$ generates the unit ideal, which is certainly open. By the discussion in §7.6, a ring of definition for $A[X]_{\{1,t\}}$ is given by $A_0[\{1,t\} \cdot X]$, which has the $J \cdot A_0[\{1,t\} \cdot X]$-adic topology, where J is an ideal of definition for A_0. Taking the quotient, we find that a ring of definition for $A(t/1)$, denoted by $A_0[t/1]$, is given by $A_0[t]$ with a certain topology on it.

With all relevant notation in place, we may proceed to simplify things dramatically by using the Tate hypothesis. Let u be a topologically nilpotent unit in A. By Proposition 7.5.3, A_0 has the u-adic topology (we may replace u by a sufficiently high power to always ensure that $u \in A_0$). Applying 7.5.3 to the localization (and noting that u is still a topologically nilpotent unit in $A(t/1)$), we find that the topology on the ring of definition $A_0[t/1] \subset A(t/1)$ is also u-adic, and $A_0[t/1][1/u]$ is equal to all of $A(t/1)$.

The same reasoning can be applied to the rings $A(1/t)$ and $A(t,1/t)$, proving that their rings of definition $A_0[t/1]$ and $A_0[t/1,1/t]$ both have the u-adic topology and inverting u generates the whole ring. It is a trivial matter to verify that 1 is an exact sequence of rings, for the underlying sequence of rings is just

$$0 \to A \to A \oplus A[1/t] \to A[1/t] \to 0.$$

The composition of the second and third maps is clearly zero, while if an element of $(r,s) \in A \oplus A[1/t]$ maps to zero then we must have $r = s$ (and in particular $s \in A$), so (r,s) comes from A.

Unfortunately, the uncompleted groups are not what we are interested in, and in general completing a sequence of topological rings does not preserve exactness (remember, the whole point is that we do not want to assume restrictive finiteness hypotheses!). Fortunately, Lemme 2 of III.2.12 in [Bou] will come to the rescue. It states that if an exact sequence of topological rings has *strict* maps, then completion preserves exactness.[6] A map of topological rings $f : B \to C$ is strict if the topology on $f(B)$ induced by B (via a quotient if the kernel is nonzero) is the same as the topology on $f(B)$ induced the subspace topology on C. It is easy to see that f is strict if and only if it is continuous and and the induced map $B \to f(B)$ is open.

In our case, the map $\delta : A(t/1) \oplus A(1/t) \to A(t/1, 1/t)$ is always strict: by translating and scaling by powers of u, it suffices to check that the image of $A_0[t/1] \oplus A_0[1/t]$ is open in $A(t/1, 1/t)$, which is the case because this image is precisely $A_0[t/1, 1/t]$. However, it may not be the case that $\epsilon : A \to A(t/1) \oplus A(1/t)$ is strict; in particular, this is where the uniformity hypothesis comes in.

Let's see what we would need in order for ϵ to be strict. Again by translating and scaling, it would suffice to show that the image of A_0 is open, which is the case if and only if it contains a set of the form $u^n(A_0[t/1] \oplus A_0[1/t])$ for some n (as these sets form a basis around zero). In other words, there must be some n such that $u^n(\phi^{-1}(A(t/1)) \cap \psi^{-1}(A(1/t)))$ lies in A_0, where $\phi : A \to A(t/1)$ and $\psi : A \to A(1/t)$ are the canonical maps.

This is proven in two stages. First, we will show that for any Tate ring, if a global section is locally integral, then it is globally power-bounded. This establishes that $\phi^{-1}(A(t/1)) \cap \psi^{-1}(A(1/t))$ is contained in A^0. Next, we use uniformity, which implies that $u^n A^0$ is contained in A_0 for some n. Putting the two together, we have strictness of 1 and therefore exactness of the completed sequence

$$0 \to \mathcal{O}_A(X) \to \mathcal{O}_A(U) \oplus \mathcal{O}_A(V) \to \mathcal{O}_A(U \cap V) \to 0. \tag{2}$$

Thus, we need the following lemma, applied to $t_1 = 1, t_2 = t$:

Lemma 13.6.1 (generalization of Lemma 3 in [BV]) *Let A be a Tate ring and let $t_1, \ldots, t_n \in A$ be elements that generate the unit ideal. Consider the localizations*

$$A_i = A\left(\frac{\{t_1\}}{t_i}, \ldots, \frac{\{t_n\}}{t_i}\right),$$

together with the canonical maps $\phi_i : A \to A_i$ and their rings of integers

$$A_{i,0} = A_0\left[\frac{\{t_1\}}{t_i}, \ldots, \frac{\{t_n\}}{t_i}\right].$$

Then if an element $r \in A$ obeys $\phi_i(r) \in A_{i,0}$ for all i, we have $r \in A^0$.

Proof. The proof goes through exactly as in [BV], except instead of taking an η in the base field (which we are not assuming to exist) such that $\eta t_i \in A_0$ for all η, we take η to be a large enough power of a topologically nilpotent unit u, and likewise for θ. ☐

[6] There is an additional technical condition: we require that there exists a countable basis of the topology around zero. This is always satisfied for Huber rings, whose topology around zero is given by powers of a given ideal.

Having proven sheafiness for the cover $\{U, V\}$ (under the hypothesis of uniformity), we wish to bootstrap to the general case (under the hypothesis of *stable* uniformity). The basic method is standard and goes back to Tate, so I will offer only a sketch.

First one deals only with $\mathcal{O}_A$ as a presheaf of abelian groups. The method to prove that it is actually a sheaf follows the classical case very closely: first one proves the sheaf property for so-called *Laurent covers*, which are covers of the form $\{U_I\}$ where I runs over all subsets of $\{1, 2, \ldots, n\}$, $t_1, t_2, \ldots, t_n$ are elements of A, and

$$U_I = \{v \in X : v(t_i) \leq 1 \text{ for } i \in I, v(t_i) \geq 1 \text{ for } i \notin I\}.$$

This is proved by induction on n; the base case $n = 1$ has been of course proven above. Then, by using a refinement lemma exactly paralleling Lemmas 8.2.2/2-4 in [BGR], one bootstraps to so-called *rational covers*, which are those considered in Lemma 13.6.1, and then to arbitrary covers by rational domains. By the discussion in Lecture 12, this suffices to show that $\mathcal{O}_A$ is a sheaf of abelian groups.

To prove that $\mathcal{O}_A$ is actually a sheaf of topological rings, by the discussion at the end of Lecture 12 we need only check for that for a cover $\{U_i\}$ of a rational domain U by rational domains, the induced map

$$\mathcal{O}_A(U) \to \prod_i \mathcal{O}_A(U_i) \tag{3}$$

is a topological embedding (i.e., strict). We can refine this to a rational cover by the same refinement lemma as before. By quasi-compactness these covers are finite. Then the same argument as in the case of the cover $\{U, V\}$, using Lemma 13.6.1 to control the integral elements in $\prod_i \mathcal{O}_A(U_i)$ by the power-bounded elements of $\mathcal{O}_A(U)$ and strict uniformity (which we now need, as we are dealing with an arbitrary rational domain U) to control the power-bounded elements of $\mathcal{O}_A(U)$ by the integral elements of $\mathcal{O}_A(U)$, we conclude that 3 is a topological embedding. This completes the proof of sheafiness.

To conclude that the higher cohomology of $\mathcal{O}_A$ vanishes on rational domains, one can use a Čech-theoretic approach that also mirrors Tate's original acyclicity theorem: we have already shown that the sequence 2 is right exact, and we can use the same refinement lemmas to reduce to this case. Note that cohomology is computed in the category of abelian sheaves, so we are free to disregard the topologies of the rings in question. For a detailed argument in a slightly restricted setting (but which is perfectly general and applies to the setting of sheafy Tate rings just as easily), see [KL, §2.4], especially Theorem 2.4.23. □

13.7 A corollary of the proof, and one more example

The following result and example are almost surely useless, as we will be dealing exclusively with sheafy rings from now on, but are at least moderately amusing.

Corollary 13.7.1 (Corollary 5 in [BV]) *Let $X = \mathrm{Spa}(A, A^+)$, where (A, A^+) is a Huber pair with A Tate. If $f \in \mathcal{O}_A(X)$ and there exists a cover of X by open sets such that $f|_{U_i} = 0$ for all i, then f is topologically nilpotent.*

Proof. As usual, it suffices to check on covers by rational domains, and by one of the refinement lemmas it suffices to check on rational covers (i.e., covers of the form described in Lemma 13.6.1). By that Lemma, an element that is locally zero is certainly globally power-bounded. Applying this to $u^{-1}f$, where u is a topologically nilpotent unit, we see that $u^{-1}f$ is power-bounded and hence $f = u \cdot u^{-1}f$ is topologically nilpotent. □

Example 13.7.2 ([BV, §4.3]) We have seen an example of a locally zero element that is globally nilpotent (but nonzero) in Example 13.5.1 above. One can also give an example of a Tate ring with a locally zero element that is not globally nilpotent (although by the above corollary, it must be topologically nilpotent). One takes

$$A = \mathbf{Q}_p[T, 1/T, Z]$$

and A_0 to be the $\mathbf{Z}_p$-subalgebra generated by the elements pT, p/T, $p^{-n}T^{a(n)}$, and $p^{-n}T^{-b(n)}$, where $a(n)$ and $b(n)$ are certain sequences that increase extremely rapidly. One topologizes A by taking the p-adic topology on A_0 as usual. It is easy enough to check that Z restricts to zero on the familiar rational subdomains U and V, and rather irritating to check that for each e there is a number $M(e)$ such that $Z^e \notin p^{M(e)}A_0$, which verifies that Z is not nilpotent in the completion of A. Details can be found in [BV].

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