Honors Math B
Quiz 1 Practice Questions

We will let F denote an arbitrary field.

1. Let $V = \mathcal{F}(\mathbf{Z}_{>0}, \mathbf{R})$ be the real vector space of all sequences of real numbers. Determine which of the following are subspaces of V. Of the subspaces, determine which are finite-dimensional. Of the finite-dimensional subspaces, determine a basis.
a) Sequences converging to zero.
b) Sequences that have infinitely many zeroes (e.g. $(1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots)$).
c) Arithmetic progressions; i.e., sequences of the form $(a, a + k, a + 2k, \ldots)$.
d) Decreasing sequences (i.e., sequences $\{a_n\}$ such that $a_n \geq a_{n+1}$ for all n).

2. Let $p(x) = x^2 + 2x - 3$, $q(x) = 2x^2 - 3x + 4$, and $r(x) = ax^2 - 1$. Considering p, q, and r as vectors in the real vector space of all functions $\mathbf{R} \to \mathbf{R}$, for what value(s) of a is $\{p, q, r\}$ linearly dependent?

3. Suppose that $\{x, y, z\}$ is a linearly independent set in a real vector space V. Prove that $\{x + y, x + z, y + z\}$ is a linearly independent set.

4. Let $V = \mathcal{F}(\mathbf{Z}_{>0}, \mathbf{R})$ be the vector space of all sequences of real numbers. Let $T : V \to V$ be the function that takes $(a_1, a_2, \ldots)$ to $(0, a_1, a_2, \ldots)$. Prove that T is linear and has a left inverse (i.e., a linear map $S : V \to V$ with $ST = \mathrm{Id}_V$), but that this left inverse is not a right inverse (i.e., $TS \neq \mathrm{Id}_V$). (We have seen that this does not happen for finite-dimensional vector spaces.)

5. Let V be the vector space consisting of real polynomials of degree three or less, and let $D : V \to V$ be the differentiation map. With respect to the usual basis $\{1, t, t^2, t^3\}$ of V, write down the matrix for $D^2 = D \circ D$ in two ways: first find the matrix for D and square it, then find the matrix for D^2 directly. Find bases for $\ker(D^2)$ and $\mathrm{im}(D^2)$.

7. Let $T : V \to V$ be a linear map from a vector space over F to itself satisfying $T^2 = T$ (i.e., $T(T(v)) = T(v)$ for all $v \in V$). Such a map is called *idempotent* or, in the context of linear algebra, a *projection*.
a) Prove that $\ker(T) \cap \mathrm{im}(T) = \{0\}$.
b) Prove that $\ker(T)$ and $\mathrm{im}(T)$ span V: for every $v \in V$, we can write $v = s + t$, where $s \in \ker(T)$ and $t \in \mathrm{im}(T)$. [HINT: Try letting $t = T(v)$.]

9. Prove that if A is an invertible square matrix with coefficients in F, then A^n is also invertible for any $n \geq 1$.

10. Calculate the inverse of the following real matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}.$$