Honors Math B
Optional last homework

A

Read Ch. 12 in Apostol.

B

For the purpose of the Apostol problems, use the "general" form of Green's theorem stated in the text (i.e., don't worry about whether a region is of graph type).

Some helpful/good problems are Apostol pp. 385-6 exercises 1ad, 2, 4, 5, 6, 7, and 8ab, p. 400 exercises 8, 9, 14, and 16abc, pp. 413-6 exercises 1, 7, 18, and 33a, p. 424 exercises 8 and 10, p. 437 exercises 5 and 7, pp. 442-443 exercises 1, 3, 6, and 11, p. 447 exercises 1a, 3, 5, and 8, pp. 452-453 exercises 3 and 10, and pp. 462-463 exercises 1 and 4-12 inclusive.

C

1. Let a, b, c, d be continuous functions $[0,1] \to \mathbf{R}$ with $a < b$ and $c < d$. Let $Q(t)$ denote the rectangle $[a(t), b(t)] \times [c(t), d(t)]$. Show that if $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous, then $\int_{Q(t)} f$ is a continuous function of t.

2. Let $f : [a, b] \to \mathbf{R}$ be a differentiable function. Express the subset of $\mathbf{R}^3$ swept out by the graph of f as it rotates about the horizontal x-axis as a parametric surface in $\mathbf{R}^3$: the *surface of revolution* associated to f. Write down an expression for the surface area between $x = a$ and $x = b$ as a one-variable integral from a to b.

3. Let $r : T \to S \subset \mathbf{R}^3$ be a parametric surface and $\gamma : [a, b] \to T$ a path in the plane, so in particular $r \circ \gamma : [a, b] \to S$ is a path in $\mathbf{R}^3$. Show that the tangent vector to $r \circ \gamma$ at each t is orthogonal to the outward normal vector of the surface.

4. Let $r : T \to S$ be a parametrized surface. Suppose that, as a set, $S = \{x \in \mathbf{R}^3 \mid f(x) = 0\}$ for some differentiable $f : \mathbf{R}^3 \to \mathbf{R}$ (we say that S is *implicitly defined*). Show that at each point of S, the gradient ∇f is a nonzero scalar multiple of the outward normal vector of the surface. Give an example.

5. Thinking in terms of physical intuition (divergence as "amount of flow created at a point"), write down an analogue of the divergence theorem in a two-dimensional region, and show that your theorem follows quickly from Green's theorem.

6. Let $U = \mathbf{R}^2 \setminus \{(0,0)\}$ be the punctured plane. We have seen that the vector field $F : U \to \mathbf{R}^2$ given by

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

is closed but not conservative: more specifically, if $\gamma : [0, 2\pi] \to U$ is the path around the unit circle given by $\gamma(t) = (\cos t, \sin t)$ then we calculated

$$\int F \cdot \mathrm{d}\gamma = 2\pi \neq 0.$$

We wish to establish that, up to scalar multiple and adding functions that are gradients, this is the *only* example of a closed vector field on U. (This calculation shows that the first *de Rham cohomology* of U is one-dimensional, which is an analytic manifestation of the fact that U has "one hole").

a) Let $U_1 = \mathbf{R}^2 \setminus \{(x,y) \mid y = 0, x \leq 0\}$ and $U_2 = \mathbf{R}^2 \setminus \{(x,y) \mid y = 0, x \geq 0\}$. Suppose $G : U \to \mathbf{R}^2$ is an arbitrary closed C^1 vector field. If $\gamma : [a,b] \to U_i$ is a closed path for $i = 1$ or $i = 2$, show that $\int G \cdot \mathrm{d}\gamma = 0$.

b) Sketch a proof, by drawing a picture and using the previous part, that if $\alpha, \beta : [a,b] \to U$ are both rectangles traversed in the counterclockwise direction whose interiors both contain the origin, then $\int G \cdot \mathrm{d}\alpha = \int G \cdot \mathrm{d}\beta$. Denote this common value by $\gamma(G)$. Show that γ is linear as a function of G.

c) Prove that if $\gamma(G) = 0$, then G is conservative. [HINT: This is the hard part. Define a scalar function ϕ via a line integral of G along rectangles, use $\gamma(G) = 0$ to show that ϕ is well-defined, and prove explicitly that $\nabla\phi = G$.]

d) Prove that if $H : U \to \mathbf{R}^2$ is an arbitrary closed C^1 vector field, then there exists a scalar function ϕ on U and a real number C such that

$$H = \nabla\phi + CF,$$

where F is as in the introduction to this problem.