Honors Math B
Homework 9

A

Read Apostol, Volume II, pp. 243-255.

B

To turn in, do Apostol p. 246 exercise 5 and p. 252 exercise 6.

To do for yourself, do Apostol p. 246 exercise 2 (at least do enough to get the general idea) and p. 252 exercises 7 and 8.

C

1. To turn in: Let $f : V \to \mathbf{R}$ be a linear map from a finite-dimensional real inner product space V to $\mathbf{R}$. Prove that there exists a unique $v \in V$ such that $f(x) = \langle x, v \rangle$ for every $x \in V$. [HINT: Pick an orthonormal basis for V and define v "component by component" so that $f(x) = \langle x, v \rangle$ whenever x is an element of your basis. Then show this works in general and is unique.]

2. To turn in: Suppose that V is a finite-dimensional *complex* inner product space. We proved in class that if $T : V \to V$ is Hermitian, then V has an orthonormal basis of T-eigenvectors (with real eigenvalues!). The converse is false, which suggests we need a more general condition on T that will be *equivalent* to some diagonalizability condition. This condition is the following: call T *normal* if $T \circ T^* = T^* \circ T$, i.e., T commutes with its adjoint. We will prove that T is normal if and only if V has an orthonormal basis of T-eigenvectors (with any eigenvalues, real or complex).
a) Prove that Hermitian, skew-Hermitian, and unitary transformations are all normal. Therefore this general spectral theorem implies several special cases.
b) By considering a diagonal matrix representation of T, prove that if V has an orthonormal basis of T-eigenvectors then T is normal.

3. To turn in: A continuation of the last problem. We will start to prove the converse part of this spectral theorem: if T is normal, then V has an orthonormal basis of T-eigenvectors.
a) Show that any linear map $T : V \to V$ can be written as $T = H + iK$, where H and K are Hermitian.
b) If T is normal, show that the H and K in the above description commute.
c) Let λ be an eigenvalue of H and E_λ the corresponding eigenspace. Show that K maps E_λ into itself, so by the already-proved Hermitian case of the spectral theorem, E_λ has an orthonormal basis of K-eigenvectors.

4. To turn in: A continuation of the last problem. We will finish the proof of the "if" part of the spectral theorem.

a) Show that two commuting Hermitian linear maps can be "simultaneously diagonalized" in that there exists an orthonormal basis of V consisting of eigenvectors for both H and K.
b) Conclude: show that V has an orthonormal basis of T-eigenvectors.

5. To turn in: Show that the product $(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ is an open set in $\mathbf{R}^n$, for any reals $a_i < b_i, 1 \leq i \leq n$.