Honors Math B
Homework 8

A

No new reading; if you want to get ahead start reading Chapter 8 of Apostol, Volume II.

B

To turn in, do the following problems in Apostol, Volume II: p. 30 exercises 2 and 5, p. 118 exercise 3, p. 126 exercise 13, and p. 141 exercises 5 and 6.

To do for yourself, do p. 30 exercises 3, 4, and 10, p. 118-119 exercises 4, 5, and 6, pp. 124-126 exercises 1, 2, 6, 8, and 14, and p. 141 exercises 2 and 3.

C

1. To turn in: Let V be a finite-dimensional inner product space (over $\mathbf{R}$ or $\mathbf{C}$) and let $U \subseteq V$ be any subspace. Show that $\dim U + \dim U^{\perp} = \dim V$.

Recall that we defined the *adjoint* of a linear map $T : V \to W$ between inner product spaces to be a map $T^* : W \to V$ such that for all $v \in V, w \in W$ we have

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle.$$

We show in class that an adjoint is always unique, and that an adjoint exists if V and W are finite-dimensional.

2. To do for yourself: Let U, V, W be inner product spaces. Do not assume that they are finite dimensional; in particular, do not assume that they have orthonormal bases. Prove the following:
a) If $S : U \to V$ is a linear map that has an adjoint $S^* : V \to U$ and c is a scalar, then $cS : U \to V$ has an adjoint and $(cS)^* = \bar{c}S^*$.
b) If $S, T : U \to V$ are linear maps that have adjoints $S^*, T^* : V \to U$ then $S + T : U \to V$ has an adjoint and $(S + T)^* = S^* + T^*$.
c) If $S : U \to V$ and $T : V \to W$ are linear maps that have adjoints $S^* : V \to U$ and $T^* : W \to V$, then $T \circ S : U \to W$ has an adjoint and $(T \circ S)^* = S^* \circ T^*$.

3. To turn in: Let U, V be inner product spaces. Do not assume that they are finite dimensional; in particular, do not assume that they have orthonormal bases. Let $S : U \to V$ be a linear map that has an adjoint S^*. Prove the following:
a) S^* itself has an adjoint and $(S^*)^* = S$.
b) $\ker S^* = (\operatorname{im} S)^{\perp}$.
c) $\ker S = (\operatorname{im} S^*)^{\perp}$.

d) If S and S^* are invertible, then S^{-1} has an adjoint and $(S^*)^{-1} = (S^{-1})^*$.

4. To turn in: Now assume that U, V are finite-dimensional inner product spaces and let $S : U \to V$ be a linear map. Prove that $\dim(\operatorname{im} S^*) = \dim(\operatorname{im} S)$. Use this and the above exercise to give a new proof that if A is a real or complex square matrix, then its row rank equals its column rank.

5. To turn in: Show that A is an orthogonal (resp. unitary) matrix if and only if its rows form an orthonormal basis for $\mathbf{R}^n$ (resp. $\mathbf{C}^n$), if and only if its columns form an orthonormal basis for $\mathbf{R}^n$ (resp. $\mathbf{C}^n$).

6. To do for yourself: Show that if A and B are orthogonal (resp. unitary) matrices, then A^{-1} and AB are both orthogonal (resp. unitary) matrices.