Honors Math B
Homework 7

A

Look through Volume I, Chapter 9 if you haven't already. Then read Apostol, Volume II, pp. 114-124 and 138-141.

B

To turn in, do the following problems in Apostol, Volume II: p. 20, exercises 3, 4, and 5 and ~~p. 30 exercises 2 and 5~~. **These last two problems will be on the next assignment**

To do for yourself, do p. 21, exercises 8, 10, and 12 and p. 20 exercises 3, 4, and 10.

C

Given any polynomial f with coefficients in a field F and any matrix A, we can define the matrix $f(A)$ in the "obvious" manner, since we know how to add and multiply matrices and multiply matrices by scalars. (For example, if $f(x) = 4x^2 + 1$, then $f(A) = 4A^2 + I$.) The following statement is called the <u>Cayley-Hamilton theorem</u>: if A is a square matrix with coefficicents in F, then $p_A(A)$ is the zero matrix. Colloquially, we can say that a matrix is a root of its own characteristic polynomial.

1. To turn in:
a) What's wrong with the following "proof" of the Cayley-Hamilton theorem: $p_A(\lambda) = \det(\lambda I - A)$, so $p_A(A) = \det(AI - A) = \det(A - A) = \det(\mathbf{0}) = 0$?
b) Prove the Cayley-Hamilton theorem for 2×2 matrices by explicit calculation.

2. To turn in:
a) Prove the Cayley-Hamilton theorem for diagonal matrices by explicit calculation.
b) Prove the Cayley-Hamilton theorem for diagonalizable matrices by reducing to the diagonal case. [HINT: You probably want to first prove that if A is similar to B and f is a polynomial, then $f(A)$ is similar to $f(B)$.]

It is possible to finish the proof of the Cayley-Hamilton theorem for real or complex matrices along these lines by using topology: the space of all diagonalizable matrices is dense in the space of all matrices, and the function taking A to $p_A(A)$ is continuous, so by the above result it must be identically zero. There are "better" proofs that work for all fields using algebraic techniques.

3. To turn in: Express the following complex numbers in the form $a + bi$, where $a, b \in \mathbf{R}$:
a) $(1 + i)^2$,

b) $1/i$,
c) $(1+i)/(1-2i)$,
d) $i^5 + i^{16}$.

4. To turn in: Let f be a polynomial with real coefficients, which we may also view as a function on $\mathbf{C}$.
a) Show that $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbf{C}$.
b) Deduce that any nonreal zeroes of f (if any exist) occur in pairs of complex conjugate numbers.
c) Deduce that if $A \in M_{n\times n}(\mathbf{R})$, any nonreal eigenvalues of A occur in complex conjugate pairs.

5. To turn in: Now do problem 8(d) on p. 113 of Apostol.

6. To do for yourself: Let V be an inner product space over $\mathbf{R}$. Prove that for all $x, y \in V$ we have

$$(x, y) = \frac{1}{4}(||x+y||^2 - ||x-y||^2).$$

In particular, if we have a norm on a real vector space that happens to come from an inner product, then we can recover the inner product from the norm using this formula.