Honors Math B
Homework 6

A

Read the rest of Apostol, Volume II, Chapter 4. Then peruse the first few sections of Volume I, Chapter 9, so as to begin to familiarize yourself with the field of complex numbers **C**.

B

To turn in, do the following problems in Apostol, Volume II: p. 101 exercises 1, 2, 4, and 6, p. 108 exercise 11, and p. 113 exercises 7 and 8. **Added 2/22: you don't have to do 8d this week; it will be on the next assignment**

To do for yourself, do p. 101 exercises 10, 11, and 12, p. 107 exercises 1, 2, and 7, and pp. 112-113 exercises 2, 4, and 6.

C

1. To turn in: For each of the following real matrices, either prove that the matrix A is not diagonalizable over the real numbers, or diagonalize it (i.e., find an invertible matrix B and a diagonal matrix D such that $A = BDB^{-1}$):

$$\text{a) } A = \begin{bmatrix} 20 & -9 \\ 30 & -13 \end{bmatrix}, \qquad \text{b) } A = \begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix}, \qquad \text{c) } A = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -5 & -4 \\ 0 & 8 & 7 \end{bmatrix}.$$

2. To turn in: If A is an upper-triangular square matrix, show that the eigenvalues of A are exactly its diagonal entries a_{ii}.

3. To turn in: let A be an $n \times n$ matrix (over an arbitrary field).
a) Prove the following statement by induction on n and expansion by minors: the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$ (using Apostol's normalization) is a degree n polynomial of the form

$$(\lambda - A_{11})(\lambda - A_{22}) \cdots (\lambda - A_{nn}) + g(\lambda),$$

where $g(\lambda)$ is a polynomial of degree at most $n - 2$.
b) Use this result to prove that $p_A(\lambda)$ is a polynomial of degree n with leading term 1, second term $-\operatorname{tr}(A)$, and constant term $(-1)^n \det(A)$.

4. To turn in: If A is a square matrix, then a *square root* of A is a matrix B such that $B^2 = A$. Square roots of matrices are rather badly behaved:

a) Prove that the real matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ has exactly two square roots.

b) Prove that the real matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no square roots.

c) Prove that the real matrix $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ has exactly four square roots (!).

d) Prove that the 2×2 zero matrix over the reals has infinitely many square roots (!!). Find them all.

5. To do for yourself: Prove that if an $n \times n$ real matrix has n distinct positive eigenvalues, it has exactly 2^n square roots. [HINT: Diagonalize!]