

# Honors Math B

## Homework 3

### A

Read Apostol, Volume I, pp. 600-613, or equivalently Volume II, pp. 54-67. Acquire Volume II if you have not already.

### B

To turn in, do the following problems in Apostol, Volume I: pp. 596-597 exercises 2 and 5. (The corresponding page number in Volume II is p. 50.)

To do for yourself, do p. 590 exercises 22, 23, 24, 25, and 27, p. 597 exercises 6, 11, 12, 13, and 19, and pp. 603-604 exercises 1, 2, 6, 7, 8, 10, 11, and 14. (Corresponding page numbers in Volume II: p. 43, pp. 50-51, and pp. 57-58.) This list is of course much too long for one week; the idea is that you should get familiar with certain computational aspects of linear algebra – writing down matrices and multiplying them, etc. – and you should do just enough problems so that you develop some fluency with this in tandem with the abstract theory.

### C

1. To turn in: Let

$$W = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n \mid \sum_{i=1}^n x_i = 0 \right\}.$$

- a) Show that  $W$  is a subspace of  $\mathbf{R}^n$ .
- b) Calculate  $\dim(W)$ .

2. To turn in: Take any finite-dimensional vector space  $V$  with  $\dim(V) = n$  and choose any basis. By the theorem proved in class, there is an isomorphism of vector spaces  $\mu : M_{n \times n} \rightarrow \mathcal{L}(V, V)$  determined by choosing the given basis on both the domain and codomain. What is  $\mu^{-1}(\text{Id}_V)$ ? (In particular, this applies to  $V = \mathbf{R}^n$  with the standard basis.) Show by example that if you (somewhat perversely) choose two different bases on the domain and codomain, this result does not hold.

3. To turn in: Give a proof or counterexample to each of the following statements, where  $A$  and  $B$  are matrices of size  $m \times n$  and  $n \times p$ , respectively, over some field.

- a) If some row of  $A$  has all entries 0, then the same is true of  $AB$ .
- b) If some column of  $A$  has all entries 0, then the same is true of  $AB$ .
- c) If some column of  $B$  has all entries 0, then the same is true of  $AB$ .
- d) If two columns of  $B$  are identical, then the same is true of  $AB$ .

4. To do for yourself: Let  $P_n$  be the vector space of polynomial functions  $\mathbf{R} \rightarrow \mathbf{R}$  of degree  $\leq n$ .
- Show that the map  $G : P_n \rightarrow \mathbf{R}^k$  given by  $G(f) = (f(1), f(2), \dots, f(k))$  is linear. Show that  $G$  is surjective when  $k \leq n + 1$ .
  - Determine the dimension of the subspace of  $P_n$  consisting of polynomials satisfying  $f(1) = f(2) = \dots = f(k) = 0$ .

5. To turn in: Let  $V$  be a vector space of dimension  $n$  over a field  $F$ .

- If  $\mathbf{v} \neq \mathbf{0} \in V$ , show that there exists a linear map  $T : V \rightarrow F$  such that  $T(\mathbf{v}) = 1$ .
- If  $W \subset V$  is a *hyperplane*, i.e. a subspace of dimension  $n - 1$ , show that there exists a linear map  $T : V \rightarrow F$  such that  $W = \ker T$ .
- More generally, if  $W \subseteq V$  is any subspace, show that there exists a  $k$  and a linear map  $T : V \rightarrow F^k$  such that  $W = \ker T$ .

6. To do for yourself: Let  $V$  be a finite-dimensional vector space and let  $S \subset V$  be a nontrivial translate of a hyperplane; i.e.,

$$S = \{u + x_0 \mid u \in U\}$$

where  $U \subset V$  is a hyperplane as in the above exercise and  $x_0$  is some vector in  $V$  *not* lying in  $U$ . Show that there exists a unique linear  $T : V \rightarrow F$  such that  $T(v) = 1$  if and only if  $v \in S$ .

7. To turn in: Let  $U$  and  $V$  be vector spaces of dimensions  $m$  and  $n$ , respectively.

- If there exists a linear surjection  $T : U \rightarrow V$ , prove that  $m \geq n$ .
- If there exists a linear injection  $T : U \rightarrow V$ , prove that  $m \leq n$ .

8. To turn in: Suppose  $A \in M_{n \times m}$  and  $B \in M_{m \times n}$  are matrices such that  $AB = I_n$ . Prove that  $m \geq n$ . [HINT: First prove the corresponding statement for linear maps.]

9. To turn in: Prove that for matrices  $A \in M_{m \times n}$ ,  $B \in M_{m \times n}$ , and  $C \in M_{n \times p}$ , we have  $(A+B)C = AC + BC$ .