Honors Math B
Homework 12

A

Read Apostol, Volume II, pp. 353-371 and 378-416, as well as the handout on continuous functions in multiple variables.

B

To turn in, do Apostol pp. 362-3 exercises 1, 9, and 14 and pp. 371-2 exercises 1, 7, and 14.

To do for yourself, do Apostol p. 362 exercise 5 and pp. 371-2 exercises 4, 5 (here $[0, \pi]$ really means $[0, \pi] \times \{0\}$), 11, and 15.

C

1. To turn in: Let's construct some counterexamples. Please prove that your examples are correct.
a) Find a bounded function $f : [0,1] \times [0,1] \to \mathbf{R}$ that is not integrable.
b) Find an integrable function $g : [0,1] \times [0,1] \to \mathbf{R}$ such that for all fixed x, $\int_0^1 g(x,y)\,\mathrm{d}y$ exists, but there exists a y such that $\int_0^1 g(x,y)\,\mathrm{d}x$ does not exist. [HINT: Take g to be a step function.]
c) Find a bounded function $h : [0,1] \times [0,1] \to \mathbf{R}$ such that for all fixed x, $\int_0^1 h(x,y)\,\mathrm{d}y$ exists, but the iterated integral $\int_0^1 \int_0^1 h(x,y)\,\mathrm{d}y\,\mathrm{d}x$ does not exist.

2. To do for yourself:
a) Prove the comparison theorem: if Q is a closed rectangle and f, g are integrable on Q with $f \leq g$, then $\int_Q f \leq \int_Q g$.
b) Same as before, but with Q replaced by an arbitrary bounded set.
c) If $S \subseteq T \subset \mathbf{R}^n$ are bounded and $f \geq 0$ is integrable on S and T, show that $\int_S f \leq \int_T f$.

3. To turn in:
a) Show that if S and T are disjoint bounded subsets of $\mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}$ a function such that $\int_S f$ and $\int_T f$ exist, then $\int_{S \cup T} f$ exists and equals the sum of the previous two integrals.
b) Same as before, but now S and T are allowed to intersect in a "slice;" i.e., $S \cap T$ is a subset of the hyperplane $x_i = c$ for some index i and some constant c. This justifies extending our integration theorems from regions of graph type to much more general regions that we can "glue together" out of regions of graph type.