CONTINUITY IN MULTIPLE VARIABLES

1. Uniform continuity

This result is very closely related to Theorem 9.10 of Apostol, which can also be used to prove our theorem that differentiation and integration can be exchanged. The proof below, however, is completely different than the one sketched in the text.

Lemma. *Let $g : U \to \mathbf{R}$ be a continuous function, where $U \subseteq \mathbf{R}^{n+1}$ is an open set containing $\{\mathbf{y}\} \times [a, b]$ for some point $\mathbf{y} \in \mathbf{R}^n$. Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t \in [a, b]$, if $||\mathbf{h} - \mathbf{y}|| < \delta$ then $|g(\mathbf{h}, t) - g(\mathbf{y}, t)| < \epsilon$.*

Proof. Fix ϵ. Consider, for $s \in [a, b]$, the statement

$$\exists \delta > 0 \mid \forall t \in [a, s], \forall x \in \mathbf{R}, ||\mathbf{h} - \mathbf{y}|| < \delta \implies |g(\mathbf{h}, t) - g(\mathbf{y}, t)| < \epsilon.$$

Let S be the set of all $s \in [a, b]$ such that the above statement holds. We are trying to prove that $b \in S$.

By continuity of g at $(\mathbf{y}, a)$, certainly $a \in S$, and S is bounded above by b, so its supremum $u := \sup S$ exists. If $u < b$, then continuity of g at $(\mathbf{y}, u)$ implies that there exists a $\delta_1 > 0$ such that $||(\mathbf{h} - \mathbf{y}, t - u)|| < \delta_1$ implies that $|g(\mathbf{h}, t) - g(\mathbf{y}, u)| < \epsilon$. In particular, if $||\mathbf{h} - y|| < \frac{\delta_1}{\sqrt{2}}$ and $|t - u| > \frac{\delta_1}{\sqrt{2}}$, then by the Pythagorean theorem we conclude that $|g(\mathbf{h}, t) - g(\mathbf{y}, u)| < \epsilon$.

By the approximation property of sup and the definition of S, there exists an $s \in (u - \frac{\delta_1}{\sqrt{2}}, u)$ and some $\delta_2 > 0$ such that for all $t \in [a, s]$ and $\mathbf{h}$ satisfying $||\mathbf{h} - \mathbf{y}|| < \delta_2$ we have $|g(\mathbf{h}, t) - g(\mathbf{y}, u)| < \epsilon$. Taking δ to be the minimum of δ_1 and δ_2 and u' to be the minimum of $u + \frac{\delta_1}{2\sqrt{2}}$ and b, we find that $|g(\mathbf{h}, t) - g(\mathbf{y}, u)| < \epsilon$ whenever $h \in [a, u']$ and $||\mathbf{h} - \mathbf{y}|| < \delta$. This contradicts $u = \sup S$, hence we must have $b = \sup S$.

A similar argument with the approximation property, with b in place of u, shows that actually $b \in S$. □

In fact there is a similar result in higher dimensions as well:

Lemma. *Let $g : U \to \mathbf{R}$ be a continuous function, where $U \subseteq \mathbf{R}^{m+n}$ is an open set containing $\{y\} \times Q$, where $\mathbf{y} \in \mathbf{R}^m$ and Q is a closed rectangle in $\mathbf{R}^n$. Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t \in Q$, if $\mathbf{h} \in \mathbf{R}^m$ is such that $||\mathbf{h} - \mathbf{y}|| < \delta$ then $|g(\mathbf{h}, t) - g(\mathbf{y}, t)| < \epsilon$.*

Sketch. Induct on the dimension n. The case $n = 1$ is the previous lemma, and the induction argument is very similar to the method of the previous proof. □

2. An application: differentiation under the integral sign

We needed the following theorem in order to show that closed implies conservative for vector fields on a star-shaped domain. It says that under fairly minimal hypotheses partial differentiation and integration *in different variables* commute.

Theorem (Differentiation under the integral sign)**.** *Let $Q \subset \mathbf{R}^n$ be a closed rectangle and $R = Q \times [a, b] \subset \mathbf{R}^{n+1}$ with coordinates $(\mathbf{x}, t)$. Let $\psi : R \to \mathbf{R}$ be a C^1 function and let $\phi : Q \to \mathbf{R}$ be defined by*

$$\phi(x) := \int_a^b \psi(\mathbf{x}, t)\, \mathrm{d}t.$$

Then for all $\mathbf{x} \in \mathrm{int}(Q)$,

$$D_k\phi(\mathbf{x}) = \int_a^b D_k\psi(\mathbf{x}, t)\, \mathrm{d}t.$$

Proof. Fix $\mathbf{x} \in \operatorname{int}(Q)$. Since $\operatorname{int}(Q)$ is open, there exists a $\delta_1 > 0$ such that if $|h| < \delta_1$ then $x + h\mathbf{e}_k \in \operatorname{int}(Q)$. We have by definition

$$D_k\phi(\mathbf{x}) = \lim_{h\to 0} \frac{\phi(\mathbf{x}+h\mathbf{e}_k) - \phi(\mathbf{x})}{h} = \lim_{h\to 0} \frac{1}{h}\int_a^b \psi(\mathbf{x}+h\mathbf{e}_k, t) - \psi(\mathbf{x}, t)\,\mathrm{d}t,$$

so by moving everything inside the integral it suffices to show that

$$\lim_{h\to 0} \frac{1}{h}\int_a^b I(h, \mathbf{x}, t)\,\mathrm{d}t = 0,$$

where

$$I(h, \mathbf{x}, t) = \psi(\mathbf{x}+h\mathbf{e}_k, t) - \psi(\mathbf{x}, t) - hD_k\psi(\mathbf{x}, t).$$

Fix an $\epsilon > 0$. By the mean value theorem applied to the function ψ restricted to line segment between $\mathbf{x}$ and $\mathbf{x}+h\mathbf{e}_k$, there exists a $\mathbf{z}$ on said line segment such that

$$I(h, \mathbf{x}, t) = h(D_k\psi(\mathbf{z}, t) - D_k\psi(\mathbf{x}, t)).$$

Note that we have very little control on $\mathbf{z}$: for example, we have no idea whether or not it is continuous as a function of t. This will turn out not to matter. Apply the uniform continuity lemma to the function

$$g(h, t) = D_k\psi(\mathbf{x}+h\mathbf{e}_k, t)$$

with $y = 0$. We recover a $\delta > 0$ such that for all $t \in [0, 1]$, $|h| < \delta$ implies

$$|D_k\psi(\mathbf{x}+h\mathbf{e}_k, t) - D_k\psi(\mathbf{x}, t)| < \frac{\epsilon}{2(b-a)}.$$

Since $||\mathbf{z} - \mathbf{x}|| < |h| < \delta$, this tells us that

$$|I(h, \mathbf{x}, t)| < \frac{|h|\epsilon}{2(b-a)}.$$

Integrating, we have

$$\frac{1}{|h|}\left|\int_a^b I(h, \mathbf{x}, t)\,\mathrm{d}t\right| \le \frac{1}{|h|}\int_a^b |I(h, \mathbf{x}, t)|\,\mathrm{d}t \le \frac{|h|}{|h|}\frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2} < \epsilon.$$

We conclude. □

We needed some sort of uniform continuity here because we needed to show that the integral of $I(h, \mathbf{x}, t)$ was small independently of the choice of t, while the definition of continuity might give us a different δ for each choice of t.

This theorem is sometimes useful in a very different context: evaluating definite integrals in one variable. Here is an example of the technique. Suppose we want to evaluate the integral

$$\int_0^1 \frac{x-1}{\log x}\,\mathrm{d}x.$$

For the moment let's pretend that we are physicists and ignore convergence issues. We define a function

$$f(t) := \int_0^1 \frac{x^t - 1}{\log x}\,\mathrm{d}x.$$

By differentiating under the integral sign, we compute

$$f'(t) = \int_0^1 x^t\,\mathrm{d}x = \left[\frac{x^{t+1}}{t+1}\right]_{t=0}^{t=1} = \frac{1}{t+1}.$$

Therefore

$$f(t) = \log(t+1) + C$$

for some constant C. But we can also compute

$$f(0) = \int_0^1 \frac{1-1}{\log x}\,\mathrm{d}x = 0$$

so $C = 0$ and $f(t) = \log(t+1)$. Plugging in $t = 1$, we find that our original integral is exactly $f(1) = \log 2$.

For this technique to work, we need a definite integral whose integrand fits in a family of functions $g(x,t)$ in another variable t such that

(1) We can compute an x-antiderivative of the t-derivative of $g(x,t)$, and
(2) We can compute a t-antiderivative of that, and
(3) We can compute $g(x,t)$ explicitly for *some* t.

This may seem like a rather special situation, but it does come up now and then.

For people who are interested in the convergence issues, we can rigorously justify everything in the following way: Since $\frac{x-1}{\log x}$ is not bounded close to zero, the integral is *defined* to be

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{x-1}{\log x} \, \mathrm{d}x.$$

Set

$$f_\epsilon(t) := \int_{\epsilon}^{1} \frac{x^t - 1}{\log x} \, \mathrm{d}x.$$

Computing as before,

$$f'_\epsilon(t) = \frac{1}{t+1} - E(\epsilon, t)$$

where the error term $E(\epsilon, t) = \frac{\epsilon^{t+1}}{t+1} \leq \epsilon$ is small. Therefore

$$f_\epsilon(t) = \log(t+1) + C_\epsilon + \int_0^t E(\epsilon, t) \, \mathrm{d}t$$

and this last term is $\leq t\epsilon$. Plugging in $t = 0$ to the original formula yields $f_\epsilon(0) = 0$ as before, so $C_\epsilon = 0$ and we get

$$|f_\epsilon(t) - \log(1+t)| \leq t\epsilon.$$

Taking $t = 1$ and taking the limit as $\epsilon \to 0$ yields the result as above.

3. The extreme value theorem

Theorem (Boundedness)**.** *Let $Q \subset \mathbf{R}^n$ be a closed rectangle. Then any continuous function $f : Q \to \mathbf{R}$ is bounded.*

Sketch. Induct on the dimension n. The base case $n = 1$ has already been shown, and the inductive step is similar: for $f : Q \times [a,b] \to \mathbf{R}$, let

$$S = \{x \in [a,b] \mid f_{[a,x] \times Q} \text{ is bounded}\}$$

and consider $\sup S$. At a key point you will need to use the uniform continuity lemma. □

See Theorem 9.8 of Apostol for a completely different proof.

Theorem (Extreme value theorem)**.** *Let $Q \subset \mathbf{R}^n$ be a closed rectangle. Then any continuous function $f : Q \to \mathbf{R}$ obtains is absolute maximum and minimum.*

Sketch. Exactly as in the one-variable case, let $M = \sup f(Q)$ and apply the boundedness theorem to $g(x) := \frac{1}{M - f(x)}$. □