A TECHNICAL LEMMA ON UNIFORM CONTINUITY

The point of the following lemma is that the choice of δ can be made once and for all, independently of t ∈ [a, b]. (If t is fixed, this is nothing more than the definition of continuity of g at the point (y, t)). This was used in the proof that we can exchange differentiation and integration in reasonable situations, which in turn was used in our proof of the Poincaré Lemma for line integrals.

This result is very closely related to Theorem 9.10 of Apostol, which can also be used to prove our theorem that differentiation and integration can be exchanged. The proof below, however, is completely different than the one sketched in the text.

Lemma 1. Let g : U → R be a continuous function, where U ⊆ R² is an open set containing {y} × [a, b], where y ∈ R. Then for all ε > 0, there exists a δ > 0 such that for all t ∈ [a, b], if |h − y| < δ then |g(h, t) − g(y, t)| < ε.

Proof. Fix ε. Consider, for s ∈ [a, b], the statement

∃δ > 0|∀t ∈ [a, s], ∀x ∈ R, |h − y| < δ ⇒ |g(h, t) − g(y, t)| < ε.

Let S be the set of all s ∈ [a, b] such that the above statement holds. We are trying to prove that b ∈ S.

By continuity of g at (y, a), certainly a ∈ S, and S is bounded above by b, so its supremum sup S exists. If sup S < b, then continuity of g at (g, u) implies that there exists a δ₁ > 0 such that |(h − y, t − u)| < δ₁ implies that |g(h, t) − g(y, u)| < ε. In particular, if |h − y| < δ₁ and |t − u| > δ₁, then by the Pythagorean theorem we conclude that |g(h, t) − g(y, u)| < ε.

By the approximation property of sup and the definition of S, there exists an s ∈ (a − δ₁/√2, b) and some δ₂ > 0 such that for all (h, t) ∈ (y − δ₂, y + δ₂) × [0, s], |g(h, t) − g(y, u)| < ε. Taking δ to be the minimum of δ₁ and δ₂ and u′ to be the minimum of u + δ₁/√2 and b, we find that |g(h, t) − g(y, u)| < ε whenever (h, t) ∈ (y − δ, y + δ) × [0, u′]. This contradicts u = sup S, hence b = sup S.

A similar argument with the approximation property, with b in place of u, shows that actually b ∈ S.

In fact there is a similar result in higher dimensions as well:

Lemma 2. Let g : U → R be a continuous function, where U ⊆ Rᵐ+ⁿ is an open set containing {y} × Q, where y ∈ Rᵐ and Q is a closed rectangle in Rⁿ. Then for all ε > 0, there exists a δ > 0 such that for all t ∈ Q, if h ∈ Rᵐ is such that |h − y| < δ then |g(h, t) − g(y, t)| < ε.

Sketch. Induct on the dimension n. The case n = 1 is the previous lemma, except that y and h are now allowed to be vectors, which does not change the argument very much. The induction argument is also very similar to the method of the previous proof.