Honors Math B
Optional homework on spectral theorems

A

Read the handout on adjoints and spectral theorems.

B

Problems in Apostol, Volume II: p. 118 exercises 3, 4, 5, and 6, pp. 124-126 exercises 1, 2, 6, 8, 13, and 14, and p. 141 exercises 2, 3, 5, and 6.

C

On this assignment, we will refine the spectral theorems proved in the handout.

1. Show that $A$ is an orthogonal (resp. unitary) matrix if and only if its rows form an orthonormal basis for $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), if and only if its columns form an orthonormal basis for $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).

2. Show that if $A$ and $B$ are orthogonal (resp. unitary) matrices, then $A^{-1}$ and $AB$ are both orthogonal (resp. unitary) matrices.

3. A linear map $T : V \to V$ on a complex finite-dimensional inner product space is called non-negative if it is self-adjoint and all its eigenvalues are positive or zero. Show, using the spectral theorem, that a nonnegative linear map has an unique nonnegative square root.

4. Suppose that $V$ is a finite-dimensional real inner product space. We proved in class that if $T : V \to V$ is symmetric, then $V$ has an orthonormal basis of real $T$-eigenvectors with real eigenvalues. Prove the converse: if $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $V$ with $T(v_j) = \lambda_j v_j$ with $\lambda_j \in \mathbb{R}$, then $T$ is symmetric. Why does your proof not work in the complex case?

5. Suppose that $V$ is a finite-dimensional complex inner product space. We proved in class that if $T : V \to V$ is Hermitian, then $V$ has an orthonormal basis of $T$-eigenvectors (with real eigenvalues!). The converse is false, which suggests we need a more general condition on $T$ that will be equivalent to some diagonalizability condition. This condition is the following: call $T$ normal if $T \circ T^* = T^* \circ T$, i.e., $T$ commutes with its adjoint. We will prove that $T$ is normal if and only if $V$ has an orthonormal basis of $T$-eigenvectors (with any eigenvalues, real or complex).
   a) Prove that Hermitian, skew-Hermitian, and unitary transformations are all normal. Therefore this general spectral theorem implies several special cases.
   b) By considering a diagonal matrix representation of $T$, prove the “only if” part of this spectral theorem.
6. A continuation of the last problem. We will start to prove the “if” part of this spectral theorem.
   a) Show that any linear map $T : V \to V$ can be written as $T = H + iK$, where $H$ and $K$ are Hermitian.
   b) If $T$ is normal, show that the $H$ and $K$ in the above description commute.
   c) Let $\lambda$ be an eigenvalue of $H$ and $E_\lambda$ the corresponding eigenspace. Show that $K$ maps $E_\lambda$ into itself, so by the already-proved Hermitian case of the spectral theorem, $E_\lambda$ has an orthonormal basis of $K$-eigenvectors.

7. A continuation of the last problem. We will finish the proof of the “if” part of the spectral theorem.
   a) Show that two commuting Hermitian linear maps can be “simultaneously diagonalized” in that there exists an orthonormal basis of $V$ consisting of eigenvectors for both $H$ and $K$.
   b) Conclude: show that $V$ has an orthonormal basis of $T$-eigenvectors.