THE FUNDAMENTAL THEOREMS IN CONTEXT

The purpose of these notes is to put some of the major results in multivariable calculus in context, with an eye towards their proper generalization.

1. Some useful terminology in linear algebra

Below, we will find ourselves considering the following situation: Suppose U, V, W are vector spaces (over $\mathbf{R}$, say, although it doesn't matter for now) and suppose we are given maps $f : U \to V$ and $g : V \to W$. We draw this situation in the following diagram:

$$U \xrightarrow{f} V \xrightarrow{g} W.$$

Definition. We say that this diagram is a *complex* if $\mathrm{im}(f) \subseteq \ker(g)$. We say that it is *exact* if $\mathrm{im}(f) = \ker(g)$.

We can unpack this a little: the diagram is a complex if and only if $g \circ f$ is the zero map, and the opposite inclusion $\ker(g) \subseteq \mathrm{im}(f)$ is the statement that for any vector $v \in V$ such that $g(v) = 0$, there is a vector $u \in U$ with $f(u) = v$.

There are a couple of special cases that are worth noting. If $U = 0 = \{\mathbf{0}\}$ is the zero vector space, so f is then necessarily the zero map taking $\mathbf{0}$ to $\mathbf{0}$, then $\mathrm{im}(f) = 0$. Therefore exactness of the diagram

$$0 \to V \xrightarrow{g} W$$

is equivalent to $\ker(g) = 0$, i.e., that g is injective.

Similarly if $W = 0$, so g is then necessarily the zero map taking all of V to $\mathbf{0}$, then $\ker(g) = V$. Therefore exactness of the diagram

$$U \xrightarrow{f} V \to 0$$

is equivalent to $\mathrm{im}(f) = V$, i.e., that f is surjective. So we see that this notion encompasses both the notions of injectivity and surjectivity for linear maps.

More generally, suppose we have a longer diagram of vector spaces and linear maps

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} V_n.$$

Such a diagram is usually called a *sequence*. We say that the sequence is a *complex* if for each $0 < i < n$, $\mathrm{im}(f_{i-1}) \subseteq \ker(f_i)$. For any $0 < i < n$, we say that this sequence is *exact at* V_i if $\mathrm{im}(f_{i-1}) = \ker(f_i)$. We say that the sequence is *exact* if for each $0 < i < n$, it is exact at V_i.

2. The fundamental theorems that we know

First, let's summarize what we know in a suggestive way, without being too concerned with the technical details. For simplicity's sake, suppose that all functions considered are smooth (infinitely differentiable) and assume that they are defined on appopriate domains.

2.1. One dimension.

In one-variable calculus, scalar fields are just the usual functions; let $\mathcal{F}$ denote a reasonable vector space of functions (for example, the set of smooth function on an interval will work nicely). We have a map diff, differentiation, taking scalar fields to scalar fields:

$$\mathcal{F} \xrightarrow{\mathrm{diff}} \mathcal{F}$$

Differentiation is a linear map. One consequence of FTC I is that this map is surjective: any (nice) function is the derivative of another function (its indefinite integral). Furthermore the map is *almost* injective: if two functions have the same derivative then they are off by a constant. In the linear algebra terminology from above, we can express this by saying that the sequence

$$0 \to \mathbf{R} \to \mathcal{F} \xrightarrow{\mathrm{diff}} \mathcal{F} \to 0 \qquad (1)$$

is exact, where the second map is the "obvious" inclusion map taking a scalar c to the constant function $x \mapsto c$. Exactness at $\mathbf{R}$ is clear (since the inclusion map is manifestly injective), exactness at the first $\mathcal{F}$ is the "almost injectivity" of differentiation, and exactness at the second $\mathcal{F}$ is the surjectivity of differentiation.

Say that we are working on an interval $I = [a, b]$. We have the usual notion of integration, taking $f \in \mathcal{F}$ to $\int_I f = \int_a^b f$. We also have the map taking $f \in \mathcal{F}$ to $f(b) - f(a)$, which we might call "evaluation at the boundary ∂I of $[a, b]$" or even more suggestively "the integral of f over ∂I" (this is a sort of zero-dimensional integral). Taking this seriously, define $\int_{\partial I} f$ to be $f(b) - f(a)$.

Coming back to Sequence (1), FTC II is then the statement that if we start with a function f in the first $\mathcal{F}$, taking the integral of f on ∂I is the same as taking the differentiation map and then taking the integral on the right side (i.e., the integral over I). In symbols,

$$\int_I \operatorname{diff}(f) = \int_{\partial I} f.$$

2.2. **Two dimensions.** Now let's go to $\mathbf{R}^2$. Let $\mathcal{SF}$ denote an appropriate space of scalar fields and let $\mathcal{VF}$ denote an appropriate space of vector spaces in the plane (for example, one can take both to be the space of smooth fields on some open set $U \subseteq \mathbf{R}^2$). We can consider the following sequence of differentiation-type maps, analogously to Sequence (1):

$$0 \to \mathbf{R} \to \mathcal{SF} \xrightarrow{\text{grad}} \mathcal{VF} \xrightarrow{\text{minicurl}} \mathcal{SF} \to 0. \tag{2}$$

As before, the map $\mathbf{R} \to \mathcal{SF}$ is the inclusion map, taking a scalar to the associated constant scalar field. This inclusion map as well as both gradient and mini-curl are linear maps. It is an easy exercise, using the equality of mixed partials, to show that this sequence is always a complex. Furthermore Sequence (2) is trivially exact at $\mathbf{R}$ because the inclusion map is injective.

If the domain of definition $U \subseteq \mathbf{R}^2$ is special in some way, then we can say more. Exactness at the first $\mathcal{SF}$ is the statement that every scalar field with zero gradient is constant. This is true whenever U is connected (you proved this whenever U is a ball in a homework assignment, and the general case is not much more difficult. If U is not connected then this is not necessarily true, because the scalar field in question could be constant on each component of U without being globally constant; i.e., the constants might not all be the same.) Exactness at $\mathcal{VF}$ holds when U is star-shaped by the Poincaré lemma: it is precisely the statement that every closed vector field (minicurl zero) is conservative (is a gradient). Finally, exactness at the second $\mathcal{SF}$ is the statement that every scalar field is the minicurl of some vector field, which is true on any domain U (exercise!). In summary, if U is star-shaped, then Sequence (2) is exact.

We can interpret the fundamental theorem of line integrals (in $\mathbf{R}^2$) and Green's theorem in parallel using Sequence (2). Let C be a curve in $\mathbf{R}^2$, and as in the one-dimensional case write $\int_{\partial C} \phi$ to mean $\phi(y) - \phi(x)$, where x and y are the endpoints of C. The FTLI is the statement that if we start with a scalar field ϕ in the first $\mathcal{SF}$, taking the integral of ϕ on ∂C is equal to taking the line integral of $\operatorname{grad}(\phi)$ on C:

$$\int_C \operatorname{grad}(\phi) = \int_{\partial C} \phi.$$

Now suppose that T is a region of graph type (or a more general region for which we can reasonably define a notion of the boundary ∂T as a parametrized curve). Then Green's theorem is the statement that if we start with a vector field F in $\mathcal{VF}$, taking the integral of F on ∂T is equal to taking the integral of $\operatorname{minicurl}(F)$ on T:

$$\int_T \operatorname{minicurl}(F) = \int_{\partial T} F.$$

2.3. **Three dimensions.** Now that we have seen how this works, we can just draw the relevant sequence for (regions in) $\mathbf{R}^3$:

$$0 \to \mathbf{R} \to \mathcal{SF} \xrightarrow{\text{grad}} \mathcal{VF} \xrightarrow{\text{curl}} \mathcal{VF} \xrightarrow{\text{div}} \mathcal{SF} \to 0. \tag{3}$$

Again, all maps are linear and it is an easy exercise using equality of mixed partials that Sequence (3) is a complex (one needs to check that both the curl of a gradient and the divergence of a curl are always zero). Exactness at $\mathbf{R}$ is again automatic.

Exactness of Sequence (3) in general is again related to properties of the domain $U \subseteq \mathbf{R}^3$. Again if U is connected then the sequence is exact at the first $\mathcal{SF}$, by the same proof. The Poincaré lemma we proved implies that Sequence (3) is exact at the first $\mathcal{VF}$ is U is star-shaped. The theorem stated in the last lecture, that a divergence-free vector field is a curl if U is star-shaped, implies exactness at the second $\mathcal{VF}$ in that case. Finally, exactness at the second $\mathcal{SF}$ is the statement that every scalar field is the divergence of some vector field, which is always true (exercise!).

As our sequence gets longer, we get more fundamental theorems. If C is a curve in $\mathbf{R}^3$ and ϕ is a scalar field, then the FTLI states that

$$\int_C \operatorname{grad}(\phi) = \int_{\partial C} \phi,$$

just as before. If S is a parametrized surface and F a vector field, Stokes's theorem states that

$$\int_S \operatorname{curl}(F) = \int_{\partial S} F.$$

Finally, if W is a region of graph type and F a vector field, the divergence theorem states that

$$\int_W \operatorname{div}(F) = \int_{\partial W} F.$$

2.4. **Summary.** To remember a good proportion of multivariable calculus, just remember the following diagram:

$$0 \to \mathbf{R} \to \mathcal{F} \xrightarrow{\text{diff}} \mathcal{F} \to 0$$

$$0 \to \mathbf{R} \to \mathcal{SF} \xrightarrow{\text{grad}} \mathcal{VF} \xrightarrow{\text{minicurl}} \mathcal{SF} \to 0$$

$$0 \to \mathbf{R} \to \mathcal{SF} \xrightarrow{\text{grad}} \mathcal{VF} \xrightarrow{\text{curl}} \mathcal{VF} \xrightarrow{\text{div}} \mathcal{SF} \to 0.$$

Each sequence is a complex. If the region in question is star-shaped, each sequence is exact. Associated to each named map, there is a fundamental theorem of the form

$$\int_R \mathrm{d}(f) = \int_{\partial R} f$$

where R is some "appropriate" region with boundary ∂R, d is the named map in question, and f is an element of the domain of d.

3. A hint of algebraic topology

On several occasions I have remarked that the *failure* of the Poincaré lemma is always related to the presence of "holes" in the domain U. If you take a class in algebraic topology, you will learn how to make this notion precise, and this observation turns into a theorem. But there's another way to develop these ideas, which is to define what a "hole" is by measuring the failure of exactness of these complexes.

How does one measure the failure of exactness of a complex? We know that $\operatorname{im}(f_i) \subseteq \ker(f_{i+1})$ for each pair of adjacent maps f_i, f_{i+1}. If these were finite-dimensional vector spaces, one possibility would be to take the difference $\dim(\ker(f_{i+1})) - \dim(\operatorname{im}(f_i))$ as a measurement of the failure of exactness at i, since this would measure the difference in size between the two vector spaces. Unfortunately, in our setting the vector spaces are essentially never finite-dimensional, so this doesn't make any sense.

The solution is a construction in linear algebra that we have not discussed. Given a subspace of a vector space, $W \subseteq V$, it is possible to define the *quotient* vector space V/W. Elements of V/W are equivalence classs of elements of V, where two elements of V are considered equivalent if their difference is an element of W.

With this notion in hand, one defines the ith *cohomology* H^i of a complex of vector spaces to be the quotient vector space

$$\mathrm{H}^i = \ker(f_i)/\operatorname{im}(f_{i-1}).$$

In the case of any of the above complexes from multivariable calculus, these quotient spaces are called the *de Rham cohomology* of the domain U. If U is star-shaped, the complexes are exact so the de Rham cohomology is trivial. In many reasonable cases, even if U is not star-shaped its ith de Rham cohomology is still finite-dimensional, and its dimension is a reasonable definition of the "number of i-dimensional holes of U" (as long as we index the maps appropriately).

In this language, the long problem on the last optional homework assignment shows that the first de Rham cohomology of the punctured plane $U = \mathbf{R}^2 \setminus \{\mathbf{0}\}$ is one-dimensional. Concretely, given a closed vector field on U, we might not be able to write it as a gradient, but we can write it in terms of a gradient using only a single choice of scalar C.

4. Generalizing to higher dimensions

The picture so far is quite compelling, but it's not at all clear how to generalize this to higher dimensions. There are two fundamental issues:

(1) What kind of thing do we integrate (functions, scalar fields, vector fields...), and how do we define the differentiation maps (like grad, curl, div...)?
(2) Over what kind of region do we integrate, and how do we define the boundary of such a region?

The ultimate answer to (2) is "oriented smooth manifold with boundary" (or, if you want to allow corners, use some geometric measure theory). We won't discuss this here at all.

Let's talk about (1). As before, assume all functions are smooth so that we can avoid unnecessary complications. For every $m \leq n$, we want a space of function-like things $\mathcal{F}_{m,n}$ that we can integrate over an m-dimensional space in some open domain $U \subseteq \mathbf{R}^n$, and for every $m < n$ we want a derivative map $\mathrm{d} : \mathcal{F}_{m,n} \to \mathcal{F}_{m+1,n}$.

For example, for every n we can let $\mathcal{F}_{0,n}$ be the space of smooth scalar fields in a domain of $\mathbf{R}^n$. Integration is just evaluation at a point. Furthermore we can let $\mathcal{F}_{1,n}$ be the space of smooth vector fields, and integration is the line integral. The derivative map

$$\mathrm{d} : \mathcal{F}_{0,n} \to \mathcal{F}_{1,n}$$

is then just the gradient.

We can also take, for every n, $\mathcal{F}_{n,n}$ to be scalar fields, $\mathcal{F}_{n-1,n}$ to be vector fields, and $\mathrm{d} : \mathcal{F}_{n-1,n} \to \mathcal{F}_{n,n}$ to be the "n-dimensional divergence" $\sum_i \frac{\partial}{\partial x_i}$. Note that we did *not* make this choice for $n = 2$ in order to get the usual form of Green's theorem and because it is not really compatible with the previous choice, but see the optional homework for an equivalent version of Green's theorem in terms of $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Integration of elements of $\mathcal{F}_{n,n}$ is the n-dimensional Riemann integral. We did not define integration of vector fields over $(n-1)$-dimensional regions in $\mathbf{R}^n$ in general, but in the case $n = 3$ this is just the surface integral. Also in the specific case $n = 3$, we've seen that we can take $\mathrm{d} : \mathcal{F}_{1,3} \to \mathcal{F}_{2,3}$ to be the curl.

In order to see how to make further definitions, let's re-examine the case $n = 3$. One way of thinking about line integrals is that we're integrating, at each point, something that "can point in the same direction as a curve" i.e. a vector. This tells us that $\mathcal{F}_{1,3}$ can be taken to be the space of vector fields. In components, if $F = (P, Q, R)$, we can write a line integral as

$$\int_C P \,\mathrm{d}x + Q \,\mathrm{d}y + R \,\mathrm{d}z$$

i.e. we can consider each coordinate direction separately and add them up. This notation is used occasionally by Apostol.

For surface integrals, we need to integrate something that can "point in the same direction as a surface" at each point. Rather than do this directly, we cheated by using the cross product, which means that instead of considering the tangent plane to a surface, we can instead consider the normal vector. But this is special to $\mathbf{R}^3$, since it is only in $\mathbf{R}^3$ that there is a nice correspondence between lines and planes (by taking orthogonal complements, say).

We didn't *have* to cheat: we could try to let $\mathcal{F}_{2,3}$ be some sort of functions on the space of "planes through the origin in $\mathbf{R}^3$ with magnitude;" i.e. it should be the space of functions on a vector space with basis the xy-plane, the yz-plane, and the xz-plane. More precisely, a function in $\mathcal{F}_{2,3}$ should be a symbol

$$P \,\mathrm{d}y \wedge \mathrm{d}z + Q \,\mathrm{d}z \wedge \mathrm{d}x + R \,\mathrm{d}x \wedge \mathrm{d}y$$

where P, Q, R are scalar functions and where, for example, $\mathrm{d}y \wedge \mathrm{d}z$ represents the yz-plane. And in fact Apostol freely uses this notation for surface integrals! So as long as we appropriately define the derivative map $\mathrm{d} : \mathcal{F}_{1,3} \to \mathcal{F}_{2,3}$ in this new context, there is no need to use the cross product at all.

Algebraically, what is going on is that $\mathcal{F}_{2,3}$ should be defined to be the space of smooth functions on a certain vector space $\Lambda^2(\mathbf{R}^3)$, called the *second exterior product of* $\mathbf{R}^3$. Since $\Lambda^2(\mathbf{R}^3)$ happens to be isomorphic to $\mathbf{R}^3$ (via the cross product!), this gives "the same" space as if we just considered vector fields.

The first genuinely new case is $\mathcal{F}_{2,4}$, which should lead to a definition of surface integrals in $\mathbf{R}^4$. Our functions should be functions on a vector space whose basis is the coordinate planes in $\mathbf{R}^4$. There are six of these: if we have coordinates (x, y, z, w), then the relevant planes are

$$\mathrm{d}x \wedge \mathrm{d}y, \quad \mathrm{d}x \wedge \mathrm{d}z, \quad \mathrm{d}x \wedge \mathrm{d}w, \quad \mathrm{d}y \wedge \mathrm{d}z, \quad \mathrm{d}y \wedge \mathrm{d}w, \quad \mathrm{d}z \wedge \mathrm{d}w.$$

This space, called $\Lambda^2(\mathbf{R}^4)$, is therefore isomorphic to $\mathbf{R}^6$. So $\mathcal{F}_{2,4}$ should at least abstractly isomorphic to the vector space of smooth functions $U \to \mathbf{R}^6$.

In general, we set $\mathcal{F}_{m,n}$ to be functions $U \to \Lambda^m(\mathbf{R}^n)$, where $\Lambda^m(\mathbf{R}^n)$ is a vector space with basis the set of coordinate m-planes in $\mathbf{R}^n$. It so happens that

$$\dim(\Lambda^m(\mathbf{R}^n)) = \binom{n}{m} = \frac{n!}{k!(n-k)!}$$

in general, so that the relevant dimensions arrange themselves into Pascal's triangle:

$$\begin{array}{ccccccccc}
 & & & \dim(\Lambda^0(\mathbf{R}^1)) = 1 & & \dim(\Lambda^1(\mathbf{R}^1)) = 1 & & & \\
 & & \dim(\Lambda^0(\mathbf{R}^2)) = 1 & & \dim(\Lambda^1(\mathbf{R}^2)) = 2 & & \dim(\Lambda^2(\mathbf{R}^2)) = 1 & & \\
 & \dim(\Lambda^0(\mathbf{R}^3)) = 1 & & \dim(\Lambda^1(\mathbf{R}^3)) = 3 & & \dim(\Lambda^2(\mathbf{R}^3)) = 3 & & \dim(\Lambda^3(\mathbf{R}^3)) = 1 & \\
\dim(\Lambda^0(\mathbf{R}^4)) = 1 & & \dim(\Lambda^1(\mathbf{R}^4)) = 4 & & \dim(\Lambda^2(\mathbf{R}^4)) = 6 & & \dim(\Lambda^3(\mathbf{R}^4)) = 4 & & \dim(\Lambda^4(\mathbf{R}^4)) = 1 \\
 & & & & \vdots & & & &
\end{array}$$

Now we can see what one needs to do in order to set up this theory properly:

(1) Define the vector spaces $\Lambda^m(\mathbf{R}^n)$, which allows us to define the spaces of functions $\mathcal{F}_{m,n}$. This is just linear algebra.
(2) Define the derivative map $\mathrm{d} : \mathcal{F}_{m,n} \to \mathcal{F}_{m+1,n}$, called the *exterior derivative*.
(3) Check that the relevant theorems still hold: the relevant sequences are still complexes, the Poincaré lemma still holds (if U is star-shaped, the complexes are exact), and the general Stokes's theorem holds.

Once the relevant technology is set up, the proof of Stokes's theorem is remarkably easy. The only analytic input necessary is the ordinary fundamental theorem of calculus. To see all this, take the *Differential Manifolds* course or read a text like Spivak, *Calculus on Manifolds*, or Madsen and Tornehave, *From Calculus to Cohomology*.