Honors Math B
Final exam practice problems

These problems vary wildly in difficulty and length, so use with caution.

1. Let S and T be linear maps $\mathbf{R}^n \to \mathbf{R}^n$ such that $S \circ T = 0$. Prove that $\operatorname{im} T \subseteq \ker S$ and that $\operatorname{rank} S + \operatorname{rank} T \leq n$.

2. Let $F : \mathbf{R}^n \to \mathbf{R}$ be a differentiable scalar-valued function and let $\mathbf{x} \in \mathbf{R}^n$ be a point and $\mathbf{h} \in \mathbf{R}^n$ a vector. Let $\phi : \mathbf{R} \to \mathbf{R}$ be the function given by

$$\phi(t) = f(\mathbf{x} + t\mathbf{h}).$$

Prove that ϕ is differentiable and

$$\phi'(t) = \nabla f(\mathbf{x} + t\mathbf{h}) \cdot \mathbf{h}.$$

Use this result to prove a mean value theorem in "arbitrary directions" in $\mathbf{R}^n$: with F, $\mathbf{x}$, and $\mathbf{h}$ as above, then there exists a number $s \in (0,1)$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \nabla f(x + s\mathbf{h}) \cdot \mathbf{h}.$$

3. Prove that any linear map $F : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable (at every point).

4. Prove that if A is a symmetric invertible matrix, then its inverse A^{-1} is symmetric.

5. State rigorously the theorem that "mixed partials are equal."

6. Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be the vector field given by $F(x, y, z) = (y + 2z, x + 2z, x + 2y)$ and let C be the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + 2y + 2z = 0$. Compute the line integral of F around C in the counterclockwise direction as seen from "above" (i.e. from a large positive z-value).

7. Let $Q = [0,1] \times [0,1] \subset \mathbf{R}^2$ and let $a, b \in [0,1]$. Construct a bounded integrable function $f : Q \to \mathbf{R}$ (depending on a, b) such that neither $\int_0^1 f(a, y)\,\mathrm{d}y$ nor $\int_0^1 f(x, b)\,\mathrm{d}x$ exist.

8. Suppose that f is a smooth scalar function on $\mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ is the function given by $g(x, y) = (x + y, xy)$. Let $h = f \circ g$ be the composition. Calculate the mixed partial derivative $D_1 D_2 h$ in terms of the partial derivatives of f.

9. Fix an integer $n > 0$. Prove that for each m, there is a function $f : \mathbf{R}^n \to \mathbf{R}^n$ that is C^m but not C^{m+1}.

10. Let V be a vector space and let S_i, $i \in I$, be a collection (possibly infinite) of subspaces of V. Prove that the intersection $\bigcap_{i \in I} S_i$ is a subspace of V.

11. Integration by parts to find antiderivatives is annoying. Usually we can instead use linear algebra.
a) Compute antiderivatives of $x^2 e^x$ and $e^x \sin(x)$ by using integration by parts (warning: this last one is a bit tricky!).
b) Let V be the span of $\{x^2 e^x, xe^x, e^x\}$ in the vector space of real-valued functions on $\mathbf{R}$. Note that the (linear) differentiation operator D maps V to itself and write down the matrix for D with respect to the given basis. Prove that D is invertible and find the inverse matrix D^{-1}. Apply this inverse matrix to the vector $(1, 0, 0)$ to find an antiderivative of $x^2 e^x$.
c) Let W be the span of $\{e^x \sin(x), e^x \cos(x)\}$ and use the same method to find an antiderivative for $e^x \sin(x)$.

12. State rigorously the theorem that "closed vector fields on nice domains are conservative."

13. Let F be a vector field on $\mathbf{R}^3$. Suppose S_1 and S_2 are two parametrized surfaces that have the same oriented boundary C, but don't intersect anywhere else. Show that

$$\int_{S_1} \operatorname{curl} F \cdot \mathrm{d}r^2 = \int_{S_2} \operatorname{curl} F \cdot \mathrm{d}r^2$$

in two ways, first using Stokes's theorem and then using the divergence theorem.

14. Write down a "random" inhomogeneous system of three linear equations in three variables (with relatively small integer coefficients, say). Solve your system by Gauss-Jordan elimination. Check your answer by substitution.