Honors Math B
Homework 2

A

Read Apostol, Volume I, pp. 585-600, or equivalently Volume II, pp. 31-54. Notation alerts: Apostol calls the kernel of a linear map $T : V \to W$ the "null space" (denoted $N(T)$) and the image the "range" (denoted $T(V)$, which makes sense).

B

To turn in, do the following problems in Apostol, Volume I: p. 561 exercise 24 and p. 582 exercise 24. (Corresponding problems in Volume II: p. 14 exercise 24 and p. 35 exercise 24.)

To do for yourself, do p. 561 exercise 22 and pp. 583 exercise 28. (Corresponding problems in Volume II: p. 14 exercise 22 and p. 36 exercise 28.)

C

1. To turn in: Let $v_1, v_2, \ldots$ be an infinite sequence of elements of V (i.e. a map $\mathbf{Z}_{>0} \to V$).
a) Prove that if $\{v_1, \ldots, v_k\}$ spans V, then so does $\{v_1, \ldots, v_n\}$ for any $n \geq k$.
b) Prove that if $\{v_1, \ldots, v_n\}$ is linearly independent, then so is $\{v_1, \ldots, v_k\}$ for any $k \leq n$.

2. To turn in: Let $T : V \to V$ be a linear map. Define T^n recursively by $T^0 = \mathrm{Id}_V$ and $T^{n+1} = T^n \circ T$.
a) If $T^3 = 0$, prove that the map $\mathrm{Id}_V - T$ is an isomorphism (a bijective linear map). [HINT: consider $\mathrm{Id}_V + T + T^2$.]
b) Generalize to the case when $T^n = 0$ for some n.

3. To turn in: In $\mathcal{F}(\mathbf{R}, \mathbf{R})$, prove that the set $\{\sin x, \sin(2x), \sin(4x), \ldots, \sin(2^n x), \ldots\}$ is linearly independent.

4. To turn in: In $\mathcal{F}(\mathbf{R}, \mathbf{R})$, prove that $\{1, 1+x, 1+x+x^2, \ldots, 1+x+x^2+\ldots+x^n, \ldots\}$ is linearly independent.

5. To turn in: Let V and W be vector spaces over the same field. On the product set $V \times W$, define addition by $(v, w) + (v', w') = (v + v', w + w')$ and scalar multiplication by $c(v, w) = (cv, cw)$.
a) Prove that with these operations, $V \times W$ is a vector space. We call it the *direct sum*, and usually write it as $V \oplus W$.
b) Prove that if V and W are finite-dimensional, then so is $V \oplus W$. State and prove a formula relating $\dim(V \oplus W)$, $\dim(V)$, and $\dim(W)$.

6. To turn in: Suppose that S is a set (possibly infinite!) and let F be a field. For each $s \in S$, define $f_s : S \to F$ to be the function such that $f_s(t) = 1$ if $s = t$ and $f_s(t) = 0$ if $s \neq t$.
a) Prove that $C = \{f_s \mid s \in S\}$ is linearly independent. Conclude that in particular the space of real sequences $\mathcal{F}(\mathbf{Z}_{>0}, \mathbf{R})$ and the space of real functions $\mathcal{F}(\mathbf{R}, \mathbf{R})$ are infinite-dimensional.
b) Suppose that S is infinite. Prove that C is *not* a basis of $\mathcal{F}(S, F)$. (We proved in class that if S is finite, then C *is* a basis.)

7. To turn in: If U, V are subspaces of a vector space X, then let $U + V$ denote the span of $U \cup V$; thus it is a subspace of X (in fact, the smallest subspace of X containing both U and V). Suppose that both U and V are finite-dimensional.
a) Show that there exist bases $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ for U and V, respectively, such that both begin with the same basis for $U \cap V$; i.e., $u_1 = v_1, \ldots, u_k = v_k$, where $k = \dim(U \cap V)$.
b) Show that in this case $u_1, \ldots, u_m, v_{k+1}, \ldots, v_n$ is a basis for $U + V$.
c) Prove that $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$.
d) Illustrate with a nontrivial example in $X = \mathbf{R}^3$.