Honors Math A
Midterm practice problems

For true/false questions, give a proof or counterexample as appropriate.

1. True/false: if $f + g$ is integrable, then one of f or g is integrable.

2. Let S be a nonempty subset of the positive integers, and set $S' = \{1 - \frac{1}{n} \mid n \in S\}$. Show that $\sup(S') < 1$ if and only if S is bounded above.

3. Let S, T be nonempty subsets of the real numbers, and suppose that
a. $\forall x \in S, y \in T, x \leq y$, and
b. $\forall \epsilon > 0, \exists x \in S$ and $y \in T$ such that $y - x \leq \epsilon$.
Prove that $\sup(S) = \inf(T)$.

4. Prove that any polynomial is integrable on any closed interval. You may use that monotonic functions are integrable.

5. True/false: if x, y are real numbers such that $y - x \geq 1$, then there is an integer n such that $x \leq n \leq y$.

6. State and prove the triangle inequality for real numbers.

7. If $f : \mathbf{R} \to \mathbf{R}$ is an even function, prove that $F(x) = \int_0^x f(t)\, \mathrm{d}t$ is an odd function.

8. True/false: if S, T are nonempty bounded-above subsets of $\mathbf{R}$ and $f : S \to T$ is a surjective function such that $f(x) < x$ for all $x \in S$, then $\sup(T) < \sup(S)$.

9. Compute $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$, with proof.

10. Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies $f(x + y) = f(x) + f(y)$ and that f is continuous at 0. Prove that f is continuous.

11. True/false: if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap C$.

12. Let $f : S \to T$ and $g : T \to U$ be functions.
a. If f and g are injective, prove that $g \circ f$ is injective.
b. If f and g are surjective, prove that $g \circ f$ is surjective.

13. Prove that if $f : [a, b] \to \mathbf{R}$ is an integrable function and $c \in \mathbf{R}$, then cf is an integrable function, and show that

$$\int_a^b (cf) = c \int_a^b f.$$

You may use the corresponding result for step functions.