Honors Math A
Homework 12

A

Read pp. 272-302 and pp. 431-443 in Apostol.

B

To turn in, do the following problems in Apostol: p. 278 exercise 5 and p. 430 exercise 17.

To do for yourself, do Apostol p. 278 exercises 3, 4, and 6 and p. 430 exercises 1, 2, 3, 6, 7, 8, and 18.

C

1. To turn in: Let $f_n : [a, b] \to \mathbf{R}$ be a sequence of integrable functions converging uniformly to $f : [a, b] \to \mathbf{R}$. Prove that f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

[HINT: You might want to use the result that states that $f : [a, b] \to \mathbf{R}$ is integrable if and only if for every $\epsilon > 0$ there exist step functions $s \leq f \leq t$ with $\int_a^b (t - s) < \epsilon$. Do not assume that the functions are continuous, since the whole point here is to relax the continuity hypothesis we assumed in class!]

2. To turn in: Let $f_n(x) = \frac{x}{1+nx^2}$.
a) Find formulas for $f(x) = \lim_{n\to\infty} f_n(x)$ and $g(x) = \lim_{n\to\infty} f_n'(x)$.
b) Prove that for all $x \in \mathbf{R}$, $|f_n(x)| \leq \sqrt{1/n}$. [HINT: Find the local extrema.] Do the f_n converge uniformly?
c) Prove that f is differentiable at every $x \in \mathbf{R}$. For what x is $f'(x) = g(x)$?

3. To do for yourself: What "theorem" is disproved by the previous exercise?

4. To do for yourself: Let $I \subseteq \mathbf{R}$ be any interval, and let $\{f_n\}$ be a sequence of bounded functions $I \to \mathbf{R}$. Prove that if $f_n \to f$ uniformly for some f, and each f_n is bounded, then the sequence is *uniformly bounded* in the sense that there exists a single $M \in \mathbf{R}$ such that for all $n \in \mathbf{Z}_{>0}$ and $x \in I$, $|f_n(x)| \leq M$.

5. To turn in: If f_n and g_n are sequences of bounded functions on an interval I, and $f_n \to f$ and $g_n \to g$ both uniformly, prove that $f_n + g_n \to f + g$ uniformly.

6. To do for yourself: With the same notation and assumptions as in the previous problem, prove that $f_n g_n \to fg$ uniformly. [HINT: Use the result of exercise 4.]

The remaining four exercises should be done in order, so you can use your previous results to solve each problem. Recall that "smooth" means "infinitely differentiable" and "analytic" means "represented by a convergent power series."

8. To do for yourself: Exercise 31 on p. 304 of Apostol.

9. To turn in:
a) Let

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that $g : \mathbf{R} \to \mathbf{R}$ is a smooth function which is positive on $(0, \infty)$. Sketch its graph.
b) With g as above, let $h(x) = g(1+x)g(1-x)$, $A = \int_{-1}^{1} h(t)\,\mathrm{d}\,t$, and $F(x) = \frac{1}{A}\int_{-1}^{x} h(t)\,\mathrm{d}\,t$. Show that $F : \mathbf{R} \to \mathbf{R}$ is a smooth function such that $F(x) = 0$ for $x < -1$ and $F(x) = 1$ for $x > 1$. Sketch its graph.
c) Show that there exists a smooth function $\phi : \mathbf{R} \to \mathbf{R}$ such that $0 \leq \phi(x) \leq 1$ for all x, $\phi(x) = 1$ for $x \in (-1, 1)$, and $\phi(x) = 0$ for $x \notin [-3, 3]$. Such a function is called a "bump function." Sketch its graph.

10. To turn in: If $g, h : (a, b) \to \mathbf{R}$ are two smooth functions and c, d are distinct points in (a, b), show that there exists $\delta > 0$ and a smooth function $f : (a, b) \to \mathbf{R}$ such that for all $x \in (a, b)$,

$$|x - c| < \delta \implies f(x) = g(x)$$

and

$$|x - d| < \delta \implies f(x) = h(x).$$

This is one indication that smooth functions are "floppy." [Hint: Use ϕ from above. You should be able to take $\delta = |c - d|/6$, for example.]

11. To turn in: Suppose that $f, g : (a, b) \to \mathbf{R}$ are two analytic functions and that there exist $c \in (a, b)$ and $\delta > 0$ such that $|x - c| < \delta$ implies $f(x) = g(x)$. Show that then $f = g$. This is one indication that analytic functions are "rigid." [HINT: Use our "scootching" method by letting $U = \{x \in [c, b) \mid f(y) = g(y) \text{ for all } y \in [c, x]\}$. Show that U has a supremum and study $f - g$ near the supremum.]