Honors Math A
Homework 11

A

Read Apostol pp. 374-427. There are quite a few topics here that we did not cover in class but which you should read about: decimal expansions, the limit comparison test, the root test, the alternating series test, Dirichlet's convergence test, rearrangement of series, and improper integrals.

B

To turn in, do the following problems in Apostol: p. 382 exercises 29 and 30, p. 391 exercise 3, and p. 411 exercise 50.

To do for yourself, do Apostol p. 382 exercises 1, 2, 31, and 32, p. 391 exercises 4, 12, and 15, and a random assortment of the series convergence exercises on p. 398, p. 402, and p. 409.

C

1. To turn in: Prove the following monotonicity property of limits of sequences: If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, and $a_n \leq b_n$ for all n, show that $A \leq B$. What can we conclude if in fact $a_n < b_n$ for all n?

2. To turn in: Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences and $M \in \mathbf{Z}_{>0}$ is a number such that $n \geq M$ implies that $a_n = b_n$. Prove that $\{a_n\}$ converges if and only if $\{b_n\}$ does, and if they both converge then they converge to the same value. (This formalizes the notion that convergence of a sequence does not depend on finitely many values.)

3. To do for yourself: Suppose $S \subseteq \mathbf{R}$. Show that $c = \sup S$ if and only if c is an upper bound for S and there exists a sequence $\{a_n\}$ with each $a_n \in S$ and $\lim_{n\to\infty} a_n = c$.

4. To turn in: Show that if a sequence $\{x_n\}$ converges, then so does $\{|x_n|\}$. Is the converse true? Give a proof or counterexample.

5. To do for yourself: If $\{a_n\}$ is a sequence indexed starting at $n = 0$, prove that for any $k \in \mathbf{Z}_{\geq 0}$, the series $\sum_{i=0}^{\infty} a_i$ converges if and only if the series $\sum_{i=k}^{\infty} a_i := \sum_{i=0}^{\infty} a_{i+k}$ does.

6. To do for yourself: Given sequences $\{a_n\}$ and $\{b_n\}$ and a number $M \in \mathbf{Z}_{>0}$ such that $n \geq M$ implies $a_n = b_n$, prove that the series $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{i=1}^{\infty} b_i$ does.

7. To turn in: Show that a function $f : \mathbf{R} \to \mathbf{R}$ is continuous if and only if whenever $\{x_n\}$ converges with $\lim_{n\to\infty} x_n = x$, then $\{f(x_n)\}$ converges with $\lim_{n\to\infty} f(x_n) = f(x)$. (We call this latter property *sequential continuity*, so this exercise proves that a function $\mathbf{R} \to \mathbf{R}$ is continuous if and only if it is sequentially continuous.)

8. To turn in:
a) Let $\{a_n\}$ be a sequence. Suppose that for all $c \in \mathbf{R}$, there exists $N \in \mathbf{Z}_{>0}$ such that for all $n \in \mathbf{Z}_{>0}$, $n \geq N$ implies $|a_n| > c$. Prove that $\{a_n\}$ is divergent. (If this condition holds, we say that $\{a_n\}$ *goes to infinity* as $n \to \infty$.)
b) Prove that if $|x| > 1$, then $\{x^n\}$ is divergent.
c) Prove that if $|x| > 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ is divergent.