Honors Math A
Homework 6

A

Read pp. 120-124 and pp. 126-141 in Apostol. This assignment will in particular use the basic properties of limits (Apostol, Theorem 3.1) and the definition of continuity: a function f is continuous at a point c if $\lim_{x\to c} f(x) = f(c)$.

B

To turn in, do the following problems in Apostol: p. 125 exercise 21 and pp. 138-139 exercises 5, 8, 21, and 31. If you give an example for 31, prove that your example is correct.

To do for yourself, do the rest of p. 138 exercises 1-14.

C

1. To turn in: Suppose that f is integable on $[a, b]$. Show that $|f|$ is integrable on $[a, b]$.

2. To turn in: For all $x \in \mathbf{R}$ and $n \in \mathbf{Z}_{\geq 0}$, define the *nth power* $x^n \in \mathbf{R}$ recursively by $x^0 = 1$ and $x^{n+1} = x^n \cdot x$.
a) Using this definition, prove that $f(x) = x^n$ is monotonic on $(-\infty, 0]$ and also on $[0, \infty)$.
b) Prove that f is integrable on any closed interval $[a, b]$.
c) Prove that any polynomial function $g(x) = \sum_{i=0}^{n} c_i x^i$, where $c_i \in \mathbf{R}$, is integrable on any $[a, b]$.

3. To turn in: Let $f : (a, b) \to \mathbf{R}$ and $x \in (a, b)$. Consider the following statements:
a) $\lim_{h\to 0} |f(x + h) - f(x)| = 0$,
b) $\lim_{h\to 0} |f(x + h) - f(x - h)| = 0$.
Show that a) implies b). Give a counterexample to show that b) need not imply a).

4. To turn in: Let $f : [0, 1] \to \mathbf{R}$ be defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/n & \text{if } x = m/n \text{ is a rational number expressed in lowest terms.} \end{cases}$$

a) Show that f is continuous at x if and only if x is irrational.
b) Show that f is integrable on $[0, 1]$ and that $\int_0^1 f(x)\,\mathrm{d}x = 0$.