Honors Math A
Final practice problems

A

For each of the following, indicate whether the statement is true or false. If true, give a proof; if false, provide a counterexample.

1. Suppose that for x in some interval $(c-R, c+R)$, we have $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} b_n(x-c)^n$. Then $a_n = b_n$ for all n.

2. There exists a function $f : \mathbf{R} \to \mathbf{R}$ that is continuous at only a single point.

3. If $\{f_n\}$ is a sequence of differentiable functions $[a,b] \to \mathbf{R}$, converging uniformly to a function f. Then f is differentiable and the sequence $\{f_n'\}$ converges uniformly to f'.

4. If $f : \mathbf{R} \to \mathbf{R}$ is a polynomial function, then f is continuous.

B

1. Prove that any function $f : \mathbf{R} \to \mathbf{R}$ can be written uniquely in the form $f = g + h$, where g is an even function and h is an odd function. (Recall that, by definition, this means that $g(-x) = g(x)$ and $h(-x) = -h(x)$ for all $x \in \mathbf{R}$.)

2. Suppose S and T are sets, and $f : S \to T$, $g : T \to S$, and $h : T \to S$ are functions. Suppose that $f \circ g = \mathrm{Id}_T$ and $h \circ f = \mathrm{Id}_S$. Prove that $g = h$.

3. Assume that f is integrable and nonnegative on $[a,b]$, and that $\int_a^b f(x)\ \mathrm{d}x = 0$. Prove that $f(x) = 0$ if f is continuous at x.

4. Suppose that $f : \mathbf{R} \to \mathbf{R}$ takes on every value exactly twice (i.e., for each $y \in \mathbf{R}$, there are exactly two points $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2) = y$). Prove that f is not continuous. [HINT: Assume f is continuous. Show that, if we fix $x_1 < x_2$ and y as above, then either $f(x) > y$ for all $x \in (x_1, x_2)$ or $f(x) < y$ for all $x \in (x_1, x_2)$. Looking outside this interval, reach a contradiction.]

5. Prove that if f and $|f|$ are both integrable on a closed interval $[a,b]$ (from a homework problem,

the assumption that $|f|$ is integrable is superfluous, but never mind that for right now), then

$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x.$$

6. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is twice differentiable with $f(0) = 0$, $f(1) = 1$, and $f'(0) = f'(1) = 0$. Prove that there is an $x \in [0, 1]$ such that $|f''(x)| \geq 2$.

7. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. You may use the fact that $\lim_{n\to\infty} \frac{\log n}{n} = 0$ (this is also a big HINT!).

8. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and let $F(x) = \int_0^x xf(t) \, \mathrm{d}t$. Determine F' in terms of f. [HINT: The answer is *not* $xf(x)$. Look carefully.]

9. Determine a formula for the function f' if f is defined as below. It is to be understood that x is restricted to those values where $f'(x)$ exists (i.e., don't worry about domains).

$$\text{a)} \quad f(x) = (2 - x^2)\cos(x^2) + 2x\sin(x), \qquad \text{b)} \quad f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}.$$

10. Determine whether each series below is convergent or divergent, and sketch a proof of the fact.

$$\text{a)} \quad \sum_{n=2}^{\infty} \frac{1}{n \log n}, \qquad \text{b)} \quad \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}, \qquad \text{c)} \quad \sum_{n=1}^{\infty} \frac{n^2}{n!}.$$

11. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function. Prove that there exists a point $c \in [0, 1]$ such that $f(c) = c$.

12. Prove that it is not possible to write $x = f(x)g(x)$, where $f, g : \mathbf{R} \to \mathbf{R}$ are differentiable functions and $f(0) = g(0) = 0$.