NOTES FOR ANALYTIC NUMBER THEORY, 3/11/2020

Reminder that Homework 6 is due the Wednesday after spring break.

So far we have shown that if χ is an even primitive Dirichlet character mod $q \geq 2$, then the completed L-function

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi)$$

satisfies the functional equation

$$\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \xi(1-s,\overline{\chi})$$

with both sides entire. Using the Euler product and this functional equation, we find that $L(s,\chi)$ has trivial zeroes at $0, -2, -4, \dots$ and the only other zeroes can be in the critical strip $0 \leq \sigma \leq 1$.

(Note that in our original definition of the completed zeta-function $\xi(s)$ we multiplied by $s(1-s)$ to cancel out the poles. Since there are no poles here we don't do that, which means that if you use this definition for the trivial character χ mod 1 then $\xi(s,\chi) = \xi(s)/(s(1-s))$.)

When χ is odd – so $\chi(-1) = -1$ – the theta function we used vanishes identically. Instead of the Gaussian e^{-cx^2}, we use (a multiple of) its derivative xe^{-cx^2}. A computation yields that if we let

$$f(x) = xe^{-\pi u (qx)^2}$$

with $u > 0$ then

$$\hat{f}(y) = \frac{iy}{(q\sqrt{u})^3} e^{-\pi y^2/(uq^2)}.$$

Therefore if we define a modified theta function by

$$\vartheta_\chi(u) := \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi n^2 u}$$

our twisted Poisson summation formula from last class yields the functional equation

$$\vartheta_\chi(u) = \frac{\tau(\chi)}{iq^2 u^{3/2}} \vartheta_\chi\left(\frac{1}{q^2 u}\right).$$

Integrating the definition of ϑ_χ and exchanging integration and summation as before, we get

$$2\pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \int_0^\infty \vartheta_\chi(u) u^{(s+1)/2} \frac{\mathrm{d}u}{u}$$

(the extra factor of u in the integral serves to cancel the extra factor of n in the functional equation for ϑ_χ). As in the even case, we find that $\vartheta_\chi(u)$ goes to 0 exponentially fast as $u \to 0^+$, so there is no problem with convergence at either end of the integral: this equation serves as an analytic continuation of $L(s,\chi)$ to the whole complex plane. Elementary manipulations as before yield that if

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi)$$

then ξ is entire and satisfies

$$\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \xi(1-s,\overline{\chi}).$$

We see that in this case, $L(s,\chi)$ has trivial zeroes at $-1, -3, -5, \dots$ and no other zeroes outside of $0 \leq \sigma \leq 1$.

One can sort of awkwardly combine the two cases into one formula: let $a = 1$ if χ is odd and $a = 0$ otherwise. Then

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi)$$

satisfies

$$\xi(s,\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}}\xi(1-s,\overline{\chi}).$$

Of course, if χ is real then $\chi = \overline{\chi}$ and we get a functional equation involving χ only – we say that χ is *self-dual*.

Define

$$\epsilon(\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}}$$

(the *epsilon factor*).

Lemma 1. *If χ is primitive, then $|\tau(\chi)| = \sqrt{q}$, so $|\epsilon(\chi)| = 1$.*

Proof. This is an application of Parseval's identity (i.e., the Pythagorean theorem) in the inner product space of functions $\mathbf{Z}/q\mathbf{Z} \to \mathbf{C}$ equipped with the inner product

$$\langle f, g\rangle := \sum_{a \bmod q} f(a)\overline{g(a)}.$$

(Note that this is the *additive group* mod q, not the multiplicative group). Then the functions $f_n(a) = e^{2\pi i n q/a}$, as n varies from 0 to $q-1$, are orthogonal and each has norm q (easy computation).

Therefore if $c : \mathbf{Z}/q\mathbf{Z}$ is *any* function we can decompose it as

$$c(n) = \frac{1}{q}\sum_{n \bmod q} \hat{c}(n) f_n(a)$$

where

$$\hat{c}(n) = \sum_{a \bmod q} c(a)\overline{f_n(a)} = \sum_{a \bmod q} c(a)e^{-2\pi i n a/q}.$$

The Pythagorean theorem then tells us that

$$\sum_{a \bmod q} |c(a)|^2 = \frac{1}{q}\sum_{n \bmod q} |\hat{c}(n)|^2.$$

Let $c = \chi$. We have seen that, since χ is primitive, $\hat{c}(n) = \tau(\chi)\overline{\chi(n)}$. Therefore the above identity reduces to

$$\phi(q) = \frac{1}{q}|\tau(\chi)|^2\phi(q) \implies |\tau(\chi)| = \sqrt{q}.$$

□

Note that if χ is a real character, then $\xi(s,\chi)$ is real-valued along the real axis, so $\epsilon(\chi) = \pm 1$ (or in other words, $\tau(\chi) = \pm i^a\sqrt{q}$). If the minus case occurs, then $L(1/2,\chi)$ is forced to be zero (and the zero must have odd multiplicity); if the plus case occurs then if $L(1/2,\chi) = 0$ the zero is forced to have even multiplicity.

In fact, it is a theorem of Gauss, which he was very proud of, that the minus case never occurs. It is conjectured that $L(1/2,\chi) > 0$ for all primitive real Dirichlet characters χ, but under the present state of knowledge all we know is that any possible vanishing there happens to even order.

Now we can really get rolling in the proof of the PNT for arithmetic progressions, since most things will be the same as for ζ. First, if we let

$$\zeta_q(s) = \prod_{\chi \bmod q} L(s,\chi)$$

then the corresponding Dirichlet series is $-\sum_{n=1}^{\infty} \Lambda_q(n)n^{-s}$ with $\Lambda_q(n) \geq 0$ for all n. This positivity is all we need to apply the $3 + 4\cos\theta + \cos 2\theta \geq 0$ trick (check!), so we conclude that $\zeta_q(s)$ is nonvanishing on $\sigma = 1$, hence each of the terms is also nonvanishing on $\sigma = 1$. By the functional equation, if χ is primitive then this also shows that $L(s,\chi)$ is nonvanishing on $\sigma = 0$ except maybe at the trivial zero when χ is even.

Define

$$\psi(x,\chi) := \sum_{n \geq x} \chi(n)\Lambda(n)$$

to be the χ version of the Chebyshev function. By the same manipulations as for the PNT, it is enough to estimate $\psi(x,\chi)$. We do this exactly as before: first use the Mellin transform to write $\psi(x,\chi)$ as an integral along a vertical line to get

$$\psi(x,\chi) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \left(-\frac{L'}{L}(s,\chi)x^s\right)\frac{\mathrm{d}s}{s} + O\left(\frac{x\log^2 x}{T}\right)$$

where $T \leq x$.

We need a partial fraction decomposition for $\frac{L'}{L}(s,\chi)$. Suppose that χ is primitive. The argument that the entire function $\xi(s,\chi)$ is of order 1 is exactly the same, so it has a Hadamard product

$$\xi(s,\chi) = Ae^{Bs}\prod_\rho (1-s/\rho)e^{s/\rho}$$

where ρ ranges over the nontrivial zeroes of $L(s,\chi)$; i.e., those zeroes with $0 < \sigma < 1$. The logarithmic derivative is therefore

$$\frac{\xi'}{\xi}(s,\chi) = B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

or, in terms of L,

$$\frac{L'}{L}(s,\chi) = B - \frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}((s+a)/2) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

We already know how to estimate $\frac{\Gamma'}{\Gamma}$, so as before the really tricky bit is the last sum. The same argument as before yields the estimate

$$\frac{L'}{L}(s,\chi) = \sum_{|\Im(s-\rho)|<1} \frac{1}{s-\rho} + O(\log|qt|)$$

for $s = \sigma + it$ where $-1 \leq \sigma \leq 2$, where the sum comprises $O(\log|qt|)$ terms. The theorem of von Mangoldt counting the number of nontrivial zeroes, proved in the same way, is

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log(qT))$$

for all, say, $T \geq 2$. Here we define $N(T,\chi)$ to be the number of nontrivial zeroes of $L(s,\chi)$ with real part $< T$ in absolute value. (Now that χ might not be real, there's no guarantee that the zeta zeroes are symmetric about the real axis, so we have to count both sides.)

If χ is not primitive, then everything we did involving the functional equation is bogus, but there's no problem for analytic purposes: if χ mod q is induced by a primitive χ' mod q' with $q'|q$, then a quick Euler product computation yields

$$L(s,\chi) = L(s,\chi')\prod_{p|q}(q - \chi'(p)p^{-s}).$$

A priori this only holds if $\sigma > 1$, since that's where the Euler products converge, but by analytic continuation it must hold everywhere. The extra factor $1 - \chi'(p)p^{-s}$ has some obvious zeroes along the imaginary line, but since we decided to count only zeroes with $0 < \sigma < 1$ they don't intervene. All in all our estimate for $N(T,\chi)$ continues to hold for nonprimitive characters, with q replaced by the conductor of χ.

By the same argument as before, we get a zero-free region for $\zeta_q(s)$: each L-function contributes $O(\log|qt|)$ and there are $\phi(q)$ of them in total so our region turns out to be

$$1 - \sigma < \frac{c}{\phi(q)\log|qt|}$$

for some c and for, say, $|t| > 2$. Whereas for the zeta function we know by direct calculation that there are no small zeroes, here that's not necessarily the case – but in any event there are only finitely many of them, so they don't affect our estimates.

Putting everything together, the same big rectangular contour integral computing the Mellin transform of the log-derivative of $L(s,\chi)$ yields

$$\psi(x,\chi) = O_\chi\left(xe^{-C_\chi\sqrt{\log x}}\right)$$

for some constant $C_\chi > 0$ – the only difference is that there is no pole at $s = 1$ so no "main term" unless $\chi = \chi_0$ in which case the above is

$$\psi(x,\chi_0) = x + O\left(xe^{-C\sqrt{\log x}}\right).$$

Taking a linear combination of the $\psi(x,\chi)$ for all χ mod q (including the trivial character and all other nonprimitive characters!) yields

$$\psi(x, a \bmod q) = \frac{1}{\phi(q)}x + O_q\left(xe^{-C_q\sqrt{\log x}}\right)$$

and hence

$$\pi(x, a \bmod q) = \frac{1}{\phi(q)}\operatorname{li}(x) + O_q\left(xe^{-C_q\sqrt{\log x}}\right).$$

This is the PNT in arithmetic progressions: every residue class mod q coprime to q has "the same" number of primes, where "the same" now means in the strong sense of natural density.

After the break, we'll discuss some results related to improving the dependence of this estimate on q: right now, we've left that completely unspecified and there are potential problems in several places (the zero-free region, the constant B).