Analytic Number Theory
Homework 9

1. The *Leech lattice* Λ is a 24-dimensional even self-dual (i.e. unimodular) lattice that can be characterized by the fact that it has no vectors of norm squared equal to 2. Granting its existence, give a formula for $r_\Lambda(m)$, the number of elements in Λ of norm squared $2m$, in terms of sums-of-powers-of-divisors functions σ_n.

Remark: The Leech lattice is an extraordinary object. Its automorphism group is, up to an order 2 central extension, one of the sporadic finite simple groups, called the *Conway group* Co_1. It has been recently proved (see `https://arxiv.org/abs/1603.06518`) that the Leech lattice yields the optimal sphere packing density in dimension 24, following a 2016 breakthrough by Maryna Viazovska that proved the same result for the E_8 lattice in dimension 8.

2. Let's tie up some loose ends from class using the Chinese Remainder Theorem:
a) Prove that if a, a', c, c' are integers with $\gcd(a, c) = 1$ and $\gcd(a', c') = 1$ that $ac' \equiv a'c$ modulo N if and only if there exists a $\lambda \in (\mathbf{Z}/N\mathbf{Z})^*$ such that $(a', c') \equiv (\lambda a, \lambda c)$.
b) Finish the proof that $[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$ for arbitrary N (we proved this in class when N was a prime power).

3. Computing directly from the definition, prove that the number of cusps of $\Gamma_0(N)$ is finite and that in fact it equals

$$\sum_{d|N} \phi(\gcd(d, N/d))$$

where ϕ is the Euler totient function.

4. The group $\Gamma_0(4)$ will be especially important for us. Draw a fundamental domain for $\Gamma_0(4)$ with connected interior, and identify the cusps.

5. Fix a congruence subgroup Γ, a positive integer N such that $T^N \in \Gamma$, and two points z_0 and z_0' in $\mathbf{Q} \cup \{\infty\}$ that represent the same cusp. Choose τ and τ' in $\Gamma(1)$ such that $\tau(\infty) = z_0$ and $\tau'(\infty) = z_0'$. If f is a modular form for Γ, we have Fourier expansions

$$f|_\tau(z) = \sum_{n=0}^{\infty} a_{\tau,n} e^{2\pi i n/N}$$

and

$$f|_{\tau'}(z) = \sum_{n=0}^{\infty} a_{\tau',n} e^{2\pi i n/N}.$$

Show that $f|_\tau(z) = f|_{\tau'}(z + j)$ for some integer j and therefore that $a_{\tau',n} = a_{\tau,n} e^{2\pi i j/N}$. In particular, $|a_{\tau',n}| = |a_{\tau,n}|$.