Analytic Number Theory
Homework 8

1. Recall that by definition $\sigma_\ell(n) = \sum_{d|n} d^\ell$, the sum of the ℓth powers of the divisors of n. Using the Fourier expansion of the Eisenstein series that we derived in class, derive an expression for $\sigma_9(n)$ in terms of $\sigma_3(m)$ and $\sigma_5(m)$ for $m \le n$. Check a few cases of your formula.

2. We can *define* the "Eisenstein series of weight 2" for $z \in \mathcal{H}$ by the Fourier expansion

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

a) Show that

$$G_2(z) = \sum_{m \neq 0} \frac{1}{m^2} + \sum_{n \neq 0} \sum_{m \in \mathbf{Z}} \frac{1}{(nz+m)^2}$$

but that the convergence is not absolute, so we cannot necessarily interchange the order or summation.

b) Consider intead the nonholomorphic sum

$$G_{2,\epsilon}(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(nz+m)^2 |nz+m|^{2\epsilon}}.$$

Show that

$$G_{2,\epsilon}\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 |cz+d|^{2\epsilon} G_{2,\epsilon}(z).$$

c) In the next problem, we will prove that $\lim_{\epsilon \to 0^+} G_{2,\epsilon}(z)$ exists and equals $G_2(z) - \pi/y$, where y is the imaginary part of z. Assuming this, prove the transformation formula

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - 2\pi i c(cz+d).$$

3. (Not to turn in; optional problem for completeness.)

a) Define a function

$$I_\epsilon(z) = 2\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(z+t)^2 |z+t|^{2\epsilon}}$$

where $z \in \mathcal{H}$ and $\epsilon > -\frac{1}{2}$. Show that

$$G_{2,\epsilon}(z) - \sum_{n=1}^{\infty} I_\epsilon(nz)$$

tends to $G_2(z)$ as $\epsilon \to 0^+$.

b) Define

$$I(\epsilon) = 2\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(t+i)^2 (t^2+1)^\epsilon}.$$

Show that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} I_\epsilon(nz) = I(\epsilon)\zeta(1+2\epsilon)/y^{1+2\epsilon}.$$

c) Evaluate $I'(0) = -2\pi$ by differentiating under the integral sign.
d) Use this to show that $\lim_{\epsilon\to 0^+} G_{2,\epsilon}(z) = G_2(z) - \pi/y$.

4. For $z \in \mathcal{H}$, define the *discriminant function*

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}.$$

Prove that $\Delta(z)$ is a cusp form of weight 12 by showing that its logarithmic derivative is (a multiple of) $G_2(z)$ and using the transformation law proved in exercise 2.

5. Use the first few Fourier coefficients and the fact that $M_{12}(\Gamma)$ is two-dimensional to prove that

$$\Delta(z) = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2).$$

(Recall that E_k is the normalization of G_k with constant Fourier coefficient 1.)