Analytic Number Theory
Homework 7

1. Recall the action of the modular group $\Gamma := \mathrm{PSL}_2(\mathbf{Z})$ on the upper half-plane $\mathcal{H}$ defined by

$$g \cdot z = \frac{az+b}{cz+d}$$

if g is represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Compute the stabilizers of this action at every point.

2. Let $\mathcal{O}(\mathcal{H})$ denote the complex vector space of holomorphic functions on the upper half-plane. If $f \in \mathcal{O}(\mathcal{H})$, define

$$(f|_k g)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

where g is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For fixed even k, check that $f \mapsto f|_k g$ defines a representation (i.e., a linear action) of $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{O}(\mathcal{H})$, and that this induces an action of Γ. Check that modular forms of weight k are exactly the fixed points of this action that are holomorphic at infinity.

3. (Double points for longer problem) There is another useful description of the Eisenstein series. If you are ever handed a representation $v \mapsto v|g$ of a group G on a vector space V and asked to find fixed points of it, one way is to start with an arbitrary $v_0 \in V$ and form the sum $v = \sum_{g \in G} v_0|g$ (if this makes sense; for example, if G is finite or if the sum converges). Furthermore if you are handed a subgroup $G_0 \subseteq G$ and a G_0-fixed vector v_0, then you can instead consider the smaller sum over cosets $v = \sum_{g \in G_0 \backslash G} v_0|g$. Apply this in the following situation: $G = \Gamma$, $V = \mathcal{O}(\mathcal{H})$, $v_0 = 1$ (the constant function), and $G_0 = \langle T \rangle$, where $T \in \Gamma$ is represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that the sum converges and is equal to the Eisenstein series considered in class, up to a constant which you should identify.