

Analytic Number Theory

Homework 6

1. Let χ be a Dirichlet character mod q and d a divisor of q . Show that χ is induced by a character mod d if and only if $\chi(a) = 1$ for all $a \in \mathbf{Z}$ with $a \equiv 1 \pmod{d}$ and $(a, q) = 1$. In particular this gives a way to compute the conductor of χ by varying over divisors of q and finding the smallest one satisfying the above condition.

2. Let q be a positive integer. Show that all Dirichlet characters mod q are real-valued if and only if q divides 24.

3. Prove that if s has real part $\sigma > 1$ then

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} < |L(s, \chi)| \leq \zeta(\sigma)$$

for all Dirichlet characters χ . Show that this is sharp: if $\sigma > 1$ and $\epsilon > 0$ then there exist infinitely many χ such that

$$L(\sigma, \chi) > \zeta(\sigma) - \epsilon$$

and infinitely many χ such that

$$L(\sigma, \chi) < \frac{\zeta(2\sigma)}{\zeta(\sigma)} + \epsilon.$$