Analytic Number Theory
Homework 5

1. Use the partial fraction decomposition of ζ'/ζ and very large rectangular contour integrals to get the exact formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}),$$

where the sum is taken symmetrically and $\psi(x)$ is defined at points of discontinuity (i.e. $x = p^k$) by the average of the upper and lower limits (i.e., at such a point $\psi(x)$ is replaced by $\psi(x) - \frac{1}{2}\Lambda(x)$). This is pretty but not as useful as the non-exact formulas with error terms that we have used so far.

2. Some definitions: suppose P is an infinite set of positive integers. A subset S of P is said to have *natural density* δ if

$$\#\{n \in S : n < x\}/\#\{n \in P : n < x\} \to \delta$$

as $x \to \infty$. Note that given the prime number theorem, the prime number theorem in arithmetic progressions is equivalent to the statement that the set of primes equal to a mod q has natural density $1/\phi(q)$. Suppose P is a set of positive integers such that $\sum_{n \in P} 1/n$ diverges. A subset S of P is said to have *logarithmic density* δ if

$$\left(\sum_{n \in S} n^{-s}\right) \Big/ \left(\sum_{n \in P} n^{-s}\right) \to \delta$$

as $s \to 1^+$.

a) If we take P to be the set of primes, show that a set S of primes has logarithmic density δ if and only if

$$\sum_{p \in S} p^{-s} \sim \delta \log \frac{1}{s-1}$$

as $s \to 1^+$. Thus Dirichlet actually proves that the set of primes equal to a mod q, if $\gcd(a,q) = 1$, has logarithmic density $1/\phi(q)$.

b) Show that if $\sum_{n \in P} 1/n$ diverges and $S \subseteq P$ has natural density δ, then it also has logarithmic density δ.

c) Show that the converse is false by letting $d \in \{1, 2, \ldots, 9\}$ and considering the set S_d of integers whose first decimal digit is d.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers with absolute values at most one, and suppose that

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$$

for all s with $\Re(s) > 1$. Show that $a_n = b_n$ for all n. Using analyticity, weaken the hypothesis as much as you can.