Analytic Number Theory
Homework 4

1. Let's fill in the details of the basic facts about the Mellin transform used in the proof of the Prime Number Theorem.
a) Compute that

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{y^s}{s}\,\mathrm{d}s = \begin{cases} 1 & y>1 \\ \frac{1}{2} & y=1 \\ 0 & y<1. \end{cases}$$

b) Prove by contour integration that for $y, c, T > 0$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s}{s}\,\mathrm{d}s = \begin{cases} 1 + O(y^c\cdot\min(1, \frac{1}{T|\log y|})) \\ O(y^c\cdot\min(1, \frac{1}{T|\log y|})). \end{cases}$$

As was briefly discussed in class, for the $O(y^c)$ part use a segment of a circle centered at the origin (to the right of the line for $y < 1$ and to the left of the line for $y > 1$) and for the other part use a big rectangle (to the right of the line for $y < 1$ and to the left of the line for $y > 1$).

2.
a) By taking the logarithmic derivative of the Hadamard product for $\Gamma(s)$, prove that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \lim_{N\to\infty}\left(\log N - \sum_{k=0}^{n}\frac{1}{s+k}\right).$$

b) Recall that for any twice differentiable function f, we derived the formula

$$\sum_{k=1}^{N} f(k) = \int_{1/2}^{N+1/2} f(y)\,\mathrm{d}y - \frac{1}{2}\int_{1/2}^{N+1/2} f''(y)\left|\left|y+\frac{1}{2}\right|\right|^2\mathrm{d}y$$

where $||x||$ is the distance from x to the nearest integer (this is a special case of the Euler-Maclaurin formulas). Use this to prove that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O_\epsilon(|s|^{-2})$$

uniformly in the region $R_\epsilon = \{s \in \mathbf{C}^* : |\Im(\log s)| < \pi - \epsilon\}$.

3. The Hadamard product for the completed zeta function is

$$\xi(s) = e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

where the product ranges over nontrivial zeta zeroes.
a) Show that $e^A = \frac{1}{2}$.

b) Show that $B = \frac{1}{2}\gamma - 1 + \frac{1}{2}\log(4\pi)$ via the following steps: using the functional equation show that

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)}.$$

Using the product formula for Γ and the series for $\log 2$, show that

$$B = \frac{1}{2}\gamma - 1 + \frac{1}{2}\log(4\pi) - \lim_{s\to 1}\left[\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right].$$

Then evaluate this last limit by using that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x - \lfloor x \rfloor)x^{-s-1}\,\mathrm{d}x.$$